THE SPACE H^p WITH $0<p<1$

STANLEY S. WALTERS

1. **Introduction.** The space H^p is defined to be the class of all functions $f \equiv f(z)$ which are regular on the interior of the unit circle and such that $\sup_{0 \leqq r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty$. For arbitrary $f \in H^p$ we then define $\|f\| \equiv \sup_{0 \leqq r<1}((1/2\pi) \int_0^{2\pi} |f(re^{i\theta})|^p d\theta)^{1/p}$, which we shall call the "norm of f." We note here that the norm so defined is not a norm in the usual sense when $0<p<1$, in view of the fact that the triangular law does not hold in this case. However, A. E. Taylor [1][1] has shown that when $1 \leqq p$, H^p is a Banach space, hence in that case it really is a norm. When $0<p<1$, H^p becomes a complete, perfectly separable, linear topological space under the topology: $U \subseteq H^p$ is open in case for arbitrary $f_0 \in U$ it is true $\exists r>0 \ni E_f[\|f-f_0\|<r]$ (a sphere of radius r about f_0) lies in U. In fact H^p is linearly homeomorphic to a closed subspace of $L^p[0, 2\pi]$, where M. M. Day [2] has defined the topology of $L^p[0, 2\pi]$. M. M. Day [2] has shown that $(L^p[0, 2\pi])^*$, the class of linear functionals on $L^p[0, 2\pi]$, is void save for the zero functional, whence one might suspect the same to be true of $(H^p)^*$, the class of linear functionals on H^p. This, however, is not the case. Defining $\gamma_{n,z}(f) \equiv f^{(n)}(z)/n!$, we show that $\gamma_{n,z} \in (H^p)^*$ for all $z \ni |z|<1$ and $n=0, 1, \cdots$. Hence, not only does H^p have nonzero functionals, but there exists a countable collection of such functionals which will distinguish members of H^p. As a result it then makes sense to speak of weak convergence in H^p, where weak convergence is defined in H^p just as in any normed linear space. It is shown that any family $\mathfrak{F} \subseteq H^p$ for which $\gamma(f)$ is bounded in $f \in \mathfrak{F}$ for each $\gamma \in (H^p)^*$ has the property that when considered as a family defined on any compact subset of the unit circle it is a uniformly bounded family. As a direct result of this, we then find that a weakly convergent sequence in H^p converges uniformly on any compact subset of the unit circle to its weak limit.

2. **Inequalities.** The following inequalities, valid for $0<p<1$, are easily established by a consideration of the function $(1+t^p)/(1+t)^p$, $t \geqq 0$:

$$(a+b)^p \leqq a^p + b^p, \qquad a \geqq 0, b \geqq 0, \tag{1}$$

$$(a^p+b^p)^{1/p} \leqq 2^{(1-p)/p}(a+b), \qquad a \geqq 0, b \geqq 0. \tag{2}$$

Presented to the Society, November 26, 1949; received by the editors August 8, 1949 and, in revised form, September 21, 1949.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

However, Theorem 19, p. 28 of [**3**] yields (1), while Theorem 13 (2.8.4), p. 24 of the same reference yields (2) by letting k and b of that reference be respectively p and unity, and the a's of that reference be a and b of (2) above. From the definition of the norm in H^p, $0<p<1$, and (1) it is then clear that if f and g are in H^p we have

$$\|f+g\|^p \leqq \|f\|^p + \|g\|^p. \tag{3}$$

Inequality (2) yields $(\|f\|^p+\|g\|^p)^{1/p} \leqq 2^{(1-p)/p}(\|f\|+\|g\|)$, whence we obtain from (3)

$$\|f+g\| \leqq 2^{(1-p)/p}(\|f\| + \|g\|). \tag{4}$$

It is to be noted that (3) insures the continuity of the norm in H^p, $0<p<1$. Unless otherwise stated, we shall henceforth assume $0<p<1$.

3. **The topology of** H^p. In the introduction it was mentioned how we topologize H^p so that it becomes a linear topological space (a vector space which is a Hausdorff space in which addition and scalar multiplication are continuous operations in the space). We shall use here those axioms set forth by W. Sierpiński [**4**] in defining a Hausdorff topological space.

THEOREM 1. *H^p is a linear topological space.*

PROOF. That H^p is a vector space is implied by either of the inequalities (3) or (4) and the obvious homogeneity of the norm. It is clear that the union of any number of open sets is open and the intersection of any two open sets is open. We note that the spheres in H^p are open sets as a direct consequence of (3). As a result of this, inequality (3) or (4), and the fact that if the norm of an element of H^p vanishes then the element is necessarily the zero element of H^p, we conclude that the Hausdorff separation axiom holds in H^p. The continuity of addition and scalar multiplication in H^p is a direct consequence of (3) and (4).

THEOREM 2. *If $f \in H^p$, then*

$$|f(z)| \leqq \frac{\|f\|}{(1-|z|)^{1/p}}, \qquad |z| < 1.$$

PROOF. We may suppose $f \neq 0$. By virtue of F. Riesz's decomposition theorem [**5**] we may write $f(z)=g(z)h(z)$, where $h(z)$ is regular and bounded by unity on $|z|<1$, and $g \in H^p$ with $\|g\|=\|f\|$ and $g(z) \neq 0$ on $|z|<1$. Thus it is clear that $[g(z)]^p$ may be defined properly so that it is a member of H^1. By Cauchy's integral formula

$$[g(z)]^p = \frac{1}{2\pi}\int_0^{2\pi} \frac{[g(\rho e^{i\theta})]^p \rho e^{i\theta} d\theta}{\rho e^{i\theta} - z}, \qquad |z| < \rho < 1.$$

Thus

$$|g(z)|^p \leqq \frac{\rho}{\rho - |z|}\left(\frac{1}{2\pi}\int_0^{2\pi} |g(\rho e^{i\theta})|^p d\theta\right)$$

$$\leqq \frac{\rho}{\rho - |z|}\cdot\|g\|^p = \frac{\rho}{\rho - |z|}\cdot\|f\|^p.$$

Thus $|f(z)|^p \leqq (1/(1-|z|))\cdot\|f\|^p$, the desired conclusion.

An arbitrary linear topological space X will be said to be complete in case every Cauchy sequence in X has a limit in X ($\{x_n\} \subseteq X$ is a Cauchy sequence in case $\lim_{n,m=\infty} (x_n - x_m) = 0$).

THEOREM 3. *H^p is complete.*

A proof of the completeness of H^p for $1 \leqq p$ is to be found in A. E. Taylor's paper [1], and the proof is trivially modified for our case when $0 < p < 1$ when we note the preceding theorem.

M. M. Day [2] has defined a topology over $L^p[0, 2\pi]$ which is equivalent to the following definition: a set $U \subseteq L^p[0, 2\pi]$ is said to be open in case for arbitrary $\phi_0 \in U$ it is true $\exists r > 0 \ni E_\phi[\|\phi - \phi_0\| < r]$ lies in U, where for arbitrary $\phi \in L^p[0, 2\pi]$, $\|\phi\| \equiv ((1/2\pi) \cdot \int_0^{2\pi} |\phi(\theta)|^p d\theta)^{1/p}$. That $L^p[0, 2\pi]$ is a linear topological space is seen just as it was that H^p is a linear topological space.

THEOREM 4. *H^p is equivalent to a closed subspace of $L^p[0, 2\pi]$ (that is, $\exists$ an algebraic isomorphism Γ of H^p onto a closed subspace of $L^p[0, 2\pi]$, the isomorphism being norm preserving).*

PROOF. F. Riesz [1] has shown that if $f \in H^p$, then $\lim_{r=1} f(re^{i\theta})$ exists almost everywhere on $[0, 2\pi]$, and moreover, if we designate $\lim_{r=1} f(re^{i\theta})$ as $f(e^{i\theta})$ we have $f(e^{i\theta}) \in L^p[0, 2\pi]$ and $\|f\| = \|f(e^{i\theta})\|$, that is, f as a member of H^p has the same norm as $f(e^{i\theta})$ has as a member of $L^p[0, 2\pi]$. We then define $\Gamma(f) \equiv f(e^{i\theta})$. That Γ is 1:1 is clear from the norm preserving property of Γ. That the range of Γ is closed is clear since a Cauchy sequence in the range of Γ maps into a Cauchy sequence in H^p.

THEOREM 5. *H^p is perfectly separable (satisfies the second axiom of countability).*

PROOF. The proof that $L^p[0, 2\pi]$, $0 < p < 1$, is separable is practically the same as the proof when $1 \leqq p < \infty$. As is the case in any metric space, separability in $L^p[0, 2\pi]$ implies perfect separability, whence every subset of $L^p[0, 2\pi]$ is likewise perfectly separable. Letting $R(\Gamma)$ be the range of Γ of Theorem 4, we see that $R(\Gamma)$ is perfectly separable. Since Γ is a homeomorphism of H^p onto $R(\Gamma)$, then H^p is perfectly separable.

4. **Linear functionals.** As in any normed linear space we have the fact that if γ is a distributive functional on H^p, then $\gamma \in (H^p)^*$ if, and only if, γ is bounded on the unit sphere in H^p. We then topologize $(H^p)^*$ just as is the conjugate space of any normed linear space, that is, we define, for arbitrary $\gamma \in (H^p)^*$, $\|\gamma\| \equiv \sup_{\|f\|=1} |\gamma(f)|$, and, as in the case of an arbitrary normed linear space, we find that $(H^p)^*$ is a Banach space.

THEOREM 6. *$\gamma_{n,z} \in (H^p)^*$, $n=0, 1, \cdots$, $|z|<1$, and*

$$\|\gamma_{0,z}\| \leqq \frac{1}{(1-|z|)^{1/p}},$$

$$\|\gamma_{n,z}\| \leqq \frac{\rho_{n,z}}{(\rho_{n,z}-|z|)^{n+1}(1-\rho_{n,z})^{1/p}}, \qquad n>0,$$

where

$$\rho_{n,z} \equiv \Big[|z|(1-p)+pn+\big\{[|z|(1-p)+pn]^2 + 4p|z|(pn+1)\big\}^{1/2}\Big]\Big/2(pn+1).$$

PROOF. We may suppose $n>0$ since the case where $n=0$ is merely a direct consequence of Theorem 2. Let $|z|<\rho<1$. Then, by Cauchy's integral formula for the derivatives of a regular function we have $\gamma_{n,z}(f)=(1/2\pi)\int_0^{2\pi} f(\rho e^{i\theta})\rho e^{i\theta}d\theta/(\rho e^{i\theta}-z)^{n+1}$, whence $|\gamma_{n,z}(f)| \leqq (\rho/(\rho-|z|)^{n+1})\cdot(\|f\|/(1-\rho)^{1/p})$, by virtue of Theorem 2. Thus it is then clear that $\gamma_{n,z}\in(H^p)^*$ and if $|z|<\rho<1$, then $\|\gamma_{n,z}\| \leqq \rho/(\rho-|z|)^{n+1}(1-\rho)^{1/p}$. The final conclusion of our theorem is then obtained by minimizing the function $\rho/(\rho-|z|)^{n+1}(1-\rho)^{1/p}$ on the interval $|z|<\rho<1$.

COROLLARY 1. *There exists a countable collection of linear functionals $\{\eta_n\}$ on H^p having the property that if $f\in H^p$ is not zero, then $\exists n \ni \eta_n(f)\neq 0$.*

PROOF. We need only choose $\eta_n \equiv \gamma_{n,z}$, where z is fixed. Another collection $\{\eta_n\}$ could be $\eta_n \equiv \gamma_{0,z_n}$, where $|z_n|<1$ and $\{z_n\}$ is a sequence of distinct points having at least one limit point in the interior of the unit circle.

THEOREM 7. *Suppose* $\mathfrak{F}$ *is a family of elements of* $H^p \ni \gamma(f)$ *is bounded on* $\mathfrak{F}$ *for each fixed* $\gamma \in (H^p)^*$. *Then* $\exists M > 0 \ni$

$$\text{(a)} \qquad |\gamma(f)| \leqq M\|\gamma\|,$$

$$\text{(b)} \qquad |f(z)| \leqq \frac{M}{(1-|z|)^{1/p}},$$

$$\text{(c)} \qquad |\gamma_{n,z}(f)| \leqq M\left\{\frac{\rho_{n,z}}{(\rho_{n,z}-|z|)^{n+1}(1-\rho)^{1/p}}\right\}, \qquad n > 0,$$

for all $f \in \mathfrak{F}$.

PROOF. In order to prove our theorem we shall have to make use of Theorem 11, p. 19 of [6], which is:

Suppose E is a metric space, $G \subseteq E$ is of second category in E, and suppose that $\Phi = \{\phi\}$ is a family of real-valued and continuous functions on E for which $\phi(x)$ is bounded above with respect to ϕ for each fixed $x \in G$. Then $\exists$ a nontrivial sphere $U \subseteq E$ on which $\{\phi(x)\}$ are bounded uniformly from above.

We shall refer to this theorem as (I). We now define $G \equiv E \equiv (H^p)^*$. Since $(H^p)^*$ is a Banach space, then $(H^p)^*$ is of second category in itself by Baire's theorem. We define $\phi_f(\gamma) \equiv |\gamma(f)|$, $f \in \mathfrak{F}$, and let $\Phi \equiv \{\phi_f\}$. That ϕ_f is continuous in γ is clear since

$$|\phi_f(\gamma) - \phi_f(\gamma')| \leqq \phi_f(\gamma - \gamma') = |\gamma(f) - \gamma'(f)| \leqq \|\gamma - \gamma'\| \cdot \|f\|.$$

We now apply (I) to obtain some $m > 0$, $r > 0$, $\bar{\gamma} \ni$ on $E_{\gamma'}[\|\gamma' - \bar{\gamma}\| \leqq r]$, $\phi_f(\gamma') \leqq m$. Thus we see that in particular for the functional $\gamma' \equiv (r\gamma/\|\gamma\|) + \bar{\gamma}$, where γ is any nonzero member of $(H^p)^*$, we have $\phi_f(\gamma') \leqq m$, or merely $|\gamma'(f)| \leqq m$, or $|(r\gamma(f)/\|\gamma\|) + \bar{\gamma}(f)| \leqq m$, or $r|\gamma(f)|/\|\gamma\| \leqq 2m$ since $\bar{\gamma} \in E_{\gamma'}[\|\gamma' - \bar{\gamma}\| \leqq r]$. Then $|\gamma(f)| \leqq (2m/r)\|\gamma\|$, and letting $M \equiv 2m/r$ we obtain conclusion (a). Conclusion (b) follows immediately from (a) by letting $\gamma \equiv \gamma_{0,z}$, and by noting Theorem 6. Conclusion (c) follows from (a) by letting $\gamma \equiv \gamma_{n,z}$ and by noting Theorem 6.

5. **Weak convergence.** We note that in view of Corollary 1 it makes sense to talk of weak convergence in H^p. As in any normed linear space we say that the sequence $\{f_n\} \subseteq H^p$ converges weakly to $f \in H^p$, written $f_n \to^w f$, in case $\lim_{n=\infty} \gamma(f_n) = \gamma(f)$ for every $\gamma \in (H^p)^*$. To be sure, Corollary 1 guarantees the uniqueness of such a weak limit.

THEOREM 8. *If* $f_n \to^w f$ *in* H^p, *then* $\lim_{n=\infty} f_n(z) = f(z)$ *uniformly on all compact subsets of the unit circle.*

PROOF. Since $\gamma(f_n) \to \gamma(f)$ for all $\gamma \in (H^p)^*$, then $\{\gamma(f_n)\}$ is bounded in n for each fixed $\gamma \in (H^p)^*$. By conclusion (b) of the previous theorem $|f_n(z)| \leqq M/(1-|z|)^{1/p}$ whence $|f_n(z)| \leqq M/(1-\rho)^{1/p}$ on $|z| \leqq \rho < 1$. Moreover $\gamma_{0,z}(f_n) \to \gamma_{0,z}(f)$, or what is the same thing $f_n(z) \to f(z)$ on $|z|<1$. Thus by Vitali's theorem $\lim_{n=\infty} f_n(z) = f(z)$ uniformly on all compact subsets of $|z|<\rho$, where $\rho<1$, and hence $\lim_{n=\infty} f_n(z) = f(z)$ uniformly on all compact subsets of $|z|<1$.

THEOREM 9. *Suppose* $\{f_n\}$ *is a bounded sequence of elements of* $H^p \ni \lim_{n=\infty} \gamma_{k,0}(f_n) = \gamma_{k,0}(f)$ *for all* $k=0, 1, \cdots$. *Then* $\lim_{n=\infty} f_n(z) = f(z)$ *uniformly on all compact subsets of the unit circle.*

PROOF. Let $\epsilon>0$ be given. Consider

$$|f_n(z) - f(z)| = \left|\sum_{k=0}^{\infty} \gamma_{k,0}(f_n - f)z^k\right| \leqq \left|\sum_{k=0}^{N} \gamma_{k,0}(f_n - f)z^k\right| + \left|\sum_{k=N+1}^{\infty} \gamma_{k,0}(f_n - f)z^k\right|, \qquad |z| \leqq \rho < 1.$$

Noting Theorem 6 we find that $\rho_{k,0} = pk/(pk+1)$, whence $\|\gamma_{k,0}\| \leqq ((pk+1)/pk)^k(pk+1)^{1/p}$. It is easily verified that $\sum_{k=1}^{\infty}((pk+1)/pk)^k \cdot (pk+1)^{1/p}\rho^k$ converges. We now choose $N \ni M\sum_{k=N+1}^{\infty}((pk+1)/pk)^k \cdot (pk+1)^{1/p}\rho^k < \epsilon/2$, where M is a bound on $\{\|f_n - f\|\}$. Thus, since $|\gamma_{k,0}(f_n - f)| \leqq \|\gamma_{k,0}\| \cdot \|f_n - f\| \leqq M((pk+1)/pk)^k(pk+1)^{1/p}$, we find that $\left|\sum_{k=N+1}^{\infty} \gamma_{k,0}(f_n - f)z^k\right| < \epsilon/2$. We now choose $\overline{N} \ni n > \overline{N}$ implies $\left|\sum_{k=0}^{N} \gamma_{k,0}(f_n - f)z^k\right| < \epsilon/2$ for all $z \ni |z| \leqq \rho$, as can well be accomplished since $\gamma_{k,0}(f_n - f) \to 0$ by hypothesis. Thus, indeed $|f_n(z) - f(z)| \leqq \epsilon/2 + \epsilon/2 = \epsilon$ on $|z| \leqq \rho$ for all $n > \overline{N}$, which concludes the proof of the theorem.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES