A NOTE ON POINTWISE CONVERGENCE

M. K. FORT, JR.

Let C be the set of all continuous real-valued functions on the closed interval $[0, 1]$. The question arises as to whether or not a metric d for C exists with the property that $\lim_{n\to\infty} d(f_n, f_0)=0$ if and only if the sequence $\{f_n\}$ converges pointwise to f_0. It is well known that such a metric d does not exist, but a correct proof of the nonexistence of such a d does not seem to be so well known.

The following false "proof" of the nonexistence of d seems to be prevalent: "The weak topology for C defines a notion of convergence which is pointwise convergence. The weak topology for C does not satisfy the first countability axiom and consequently is not metrizable. Hence no metric for C exists which defines convergence to be pointwise convergence." The fallacy in this line of reasoning lies in the fact that two topologies, one metrizable and the other nonmetrizable, can induce the same notion of convergence for sequences.[1]

We now construct a double sequence $\{f_{n,m}\}$ of elements of C. This double sequence is used to give a simple proof of the nonexistence of a metric d of the type described above.

By a *normal subdivision* of a closed interval $[a, b]$ we mean the subdivision formed by inserting the points of the sequence $x_1 =(a+b)/2$, $x_2=(x_1+b)/2$, $x_3=(x_2+b)/2, \cdots$. If n is a positive integer and $\epsilon>0$, then we say that a function f is of *type* (n, ϵ) on $[a, b]$ if the domain of f contains $[a, b]$ and if the graph of $f|[a, b]$ consists of the broken line which joins successively the points with coordinates $(a, 0)$, $(x_n, 0)$, (x_{n+1}, ϵ), (x_{n+2}, ϵ), $(x_{n+3}, 0)$, and $(b, 0)$.

We now define S_1 to be the normal subdivision of $[0, 1]$. If a subdivision S_n of $[0, 1]$ has been defined, we define S_{n+1} to be the refinement of S_n which is obtained by making a normal subdivision of each interval of S_n.

We now let T be the collection of all intervals J such that for some n, J is an interval of the subdivision S_n. T is countable, and hence we may let k be a one-to-one function whose domain is T and whose range is the set of all positive integers.

Let n and m be positive integers. We now define the function $f_{n,m}$. If J is an interval of the subdivision S_n, we define $\epsilon_J=1$ if $k(J) \leq m$ and $\epsilon_J=1/k(J)$ if $k(J)>m$. Let $f_{n,m}$ be the function on $[0, 1]$

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[1] This fallacy was pointed out to the author by R. E. Priest.

which is of type (m, ϵ_J) on each interval J of S_n. The graph of $f_{n,m}$ has a hump on each interval of S_n. It is easily seen that $f_{n,m}$ is continuous, however, since if $\delta > 0$ then the graph of $f_{n,m}$ can contain at most a finite number of humps of height greater than δ.

Let n be a positive integer and suppose $0 \leqq t \leqq 1$. It is easy to see that $f_{n,m}(t) = 0$ for all but at most three values of m. Thus, we obtain $\lim_{m \to \infty} f_{n,m}(t) = 0$.

Now let f_0 be the function which is identically zero on $[0, 1]$. Suppose a metric d exists which defines convergence to be pointwise convergence. We obtain $\lim_{m \to \infty} d(f_{n,m}, f_0) = 0$ for each positive integer n. It is possible to choose integers N_n such that if $m_n > N_n$ then $d(f_{n,m_n}, f_0) < 1/n$. It follows that any such sequence $\{f_{n,m_n}\}$ converges pointwise to f_0. We obtain a contradiction by showing that this is not the case.

Let J_1 be an interval of S_1 and choose $m_1 > \max(k(J_1), N_1)$. We see that f_{1,m_1} is 1 on some interval J_2 of S_2, $J_2 \subset J_1$. Next choose $m_2 > \max(k(J_2), N_2)$. It follows that f_{2,m_2} is 1 on some interval J_3 of S_3, $J_3 \subset J_2$. Then choose $m_3 > \max(k(J_3), N_3)$. The procedure is clear. We obtain a nested sequence $J_1 \supset J_2 \supset J_3 \supset \cdots$ of closed intervals and integers $m_n > N_n$ such that f_{n,m_n} is 1 on J_{n+1}. There is a point p which belongs to each interval J_n. We now obtain

$$\lim_{n \to \infty} f_{n,m_n}(p) = 1 \neq 0 = f_0(p).$$

This proves that $\{f_{n,m_n}\}$ does not converge pointwise to f_0.

UNIVERSITY OF ILLINOIS