ON THE WALSH-KACZMARZ SERIES

GEN-ICHIRÔ SUNOUCHI

1. Let $\{\psi_n(t)\}$ be the Walsh-Kaczmarz functions, which form a normalized set of orthogonal functions. Every function $f(t)$ integrable in the interval (0, 1) can be expanded by means of the functions $\psi_n(t)$ in the form

$$f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t), \tag{1}$$

where c_n are defined by the equation $c_n = \int_0^1 f(t)\psi_n(t)dt$. We term (1) the Walsh-Kaczmarz series of $f(t)$. By $s_n(t)$ and $\sigma_n(t)$ we denote the nth partial sum and the nth arithmetic mean of the series (1), respectively. We prove the following theorems.

THEOREM 1. *If $s_{n,\nu}(t)$ denotes the νth partial sum of the Walsh-Kaczmarz series of $f_n(t)$, then*

$$\int_0^1 \left(\sum_{n=0}^{\infty} | s_{n,p_n}(t) |^2 \right)^{r/2} dt \leqq A_r \int_0^1 \left(\sum_{n=0}^{\infty} | f_n(t) |^2 \right)^{r/2} dt,^1 \qquad r > 1.$$

THEOREM 2. *If $f(t)$ belongs to L^r $(r>1)$, then*

$$B_r \int_0^1 \left(\sum_{n=1}^{\infty} \frac{| s_n(t) - \sigma_n(t) |^2}{n} \right)^{r/2} dt \leqq \int_0^1 | f(t) |^r dt \leqq C_r \int_0^1 \left(\sum_{n=1}^{\infty} \frac{| s_n(t) - \sigma_n(t) |^2}{n} \right)^{r/2} dt.$$

THEOREM 3. *If $f(t)$ belongs to L^r $(r>1)$, then*

$$D_r \int_0^1 \left(\sum_{n=1}^{\infty} | s_{2^n}(t) - \sigma_{2^n}(t) |^2 \right)^{r/2} dt \leqq \int_0^1 | f(t) |^r dt \leqq E_r \int_0^1 \left(\sum_{n=1}^{\infty} | s_{2^n}(t) - \sigma_{2^n}(t) |^2 \right)^{r/2} dt.$$

THEOREM 4. *If $f(t)$ belongs to L^r $(r>1)$, then*

$$\int_0^1 \left(\sup_{0 \leqq n < \infty} | \sigma_n(t) | \right)^r dt \leqq F_r \int_0^1 | f(t) |^r dt.$$

Received by the editors May 14, 1949 and, in revised form, October 7, 1949.

[1] $A_r, B_r, \cdots$ denote constants depending only on r.

THEOREM 5. *If $f(t) \in L^r$ $(r>1)$ and $2 \geqq k > 1$, then*

$$\int_0^1 \left(\sup_{0 \leqq n < \infty} \frac{1}{n+1} \sum_{\nu=0}^{n} | s_\nu(t) - f(t) |^k \right)^{r/k} dt \leqq G_r \int_0^1 | f(t) |^r dt.$$

In particular

$$\frac{1}{n+1} \sum_{\nu=0}^{n} | s_\nu(t) - f(t) |^2$$

converges to zero p.p. in $(0, 1)$.

THEOREM 6. *If $f(t)$ belongs to L^r $(r>1)$, then for almost all t the sequence of natural numbers can be divided into two complementary subsequences $\{\nu_k\}$ and $\{\mu_{k},\}$ such that $s_{\nu_k}(t)$ tends to $f(t)$ and $\sum 1/\mu_k$ converges.*

Theorem 4 is a special case of a theorem due to Paley [3][2] which is the maximal theorem for $\sigma_n^\delta(t)$ $(\delta>0)$, $\sigma_n^\delta(t)$ being the nth Cesàro mean of the δth order of the series (1). But the complete proof of the latter theorem seems not yet to be published.

Theorem 5 was also stated without proof by Paley [3].

2. LEMMA 1. *If $f(t)$ belongs to L^r $(r>1)$, then*

$$A_r \int_0^1 \left(\sum_{n=0}^{\infty} | \Delta_n(t) |^2 \right)^{r/2} dt \leqq \int_0^1 | f(t) |^r dt \leqq B_r \int_0^1 \left(\sum_{n=0}^{\infty} | \Delta_n(t) |^2 \right)^{r/2} dt,$$

where

$$\Delta_\nu(t) = \sum_{n=2^\nu}^{2^{\nu+1}-1} c_n \psi_n(t).$$

This is due to Paley [2].

LEMMA 2. *Let*

$$f_n(t) \sim \sum_{\nu=0}^{\infty} c_{n,\nu} \psi_\nu(t),$$

$$\Delta_\nu(f_n) = \sum_{i=2^\nu}^{2^{\nu+1}-1} c_{n,i} \psi_i(t),$$

then

[2] Numbers in brackets refer to the bibliography at the end of the paper.

$$C_r \int_0^1 \left(\sum_{n,\nu} |\Delta_\nu(f_n)|^2 \right)^{r/2} dt \leqq \int_0^1 \left(\sum_n f_n^2(t) \right)^{r/2} dt \leqq D_r \int_0^1 \left(\sum_{n,\nu} |\Delta_\nu(f_n)|^2 \right)^{r/2} dt,$$

where $r>1$.

This is a generalization of Lemma 1. Marcinkiewicz [1] proved this in the trigonometrical case, where the proof depends on the trigonometrical analogue of Lemma 1. Following his argument and using Lemma 1, we can prove Lemma 2.

We shall now prove Theorem 1.[3] Let us put

$$k = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_\lambda}, \qquad g(t) = f(t)\cdot\psi_k(t) \sim \sum_{m=0}^{\infty} c'_m \psi_m(t),$$

and

$$g_k(t) = \sum_{m=2^k}^{2^{k+1}-1} c'_m \psi_m(t).$$

For

$$m = 2^{m_1} + 2^{m_2} + \cdots, \quad 0 \leqq m_1 < m_2 < \cdots,$$
$$k = 2^{k_1} + 2^{k_2} + \cdots, \quad 0 \leqq k_1 < k_2 < \cdots,$$

define

$$m \dotplus k = \sum_{m_i \neq k_j} 2^{m_i} + \sum_{k_j \neq m_i} 2^{k_j}, \qquad 0 \dotplus k = k \dotplus 0 = k.$$

Then $\psi_m(t)\psi_k(t) = \psi_{m \dotplus k}(t)$, so

$$\psi_k(t) f(t) \sim \sum_n c_n \psi_n(t)\psi_k(t) = \sum_n c_n \psi_{n \dotplus k}(t) = \sum_m c_{m \dotplus k} \psi_m(t),$$

for $m = n \dotplus k$ implies $n = m \dotplus k$. Hence $c'_m = c_{m \dotplus k}$.

Now

$$\sum_{r=1}^{\lambda} g_{k_r}(t) = \sum_{r=1}^{\lambda} \sum_{m=2^{k_r}}^{2^{k_r+1}-1} c'_m \psi_m(t)$$
$$= \sum_{r=1}^{\lambda} \sum_{m=2^{k_r}}^{2^{k_r+1}-1} \psi_k(t) c_{m \dotplus k} \psi_{m \dotplus k}(t)$$
$$= \psi_k(t) \sum_{n \in A} c_n \psi_n(t)$$

[3] The author has to thank the referee for his valuable suggestions for this proof.

where A is the set of indices, k in number,

$$m \dotplus k \qquad (2^{k_r} \leqq m < 2^{k_r+1};\ r = 1, 2, \cdots, \lambda).$$

No two indices are equal, since the non-negative integers form a group under $\dotplus$. Also $m \dotplus k < k$, since

$$m = 2^{k_r} \dotplus \mu \ (0 \leqq \mu < 2^{k_r}), \qquad k = 2^{k_r} \dotplus (k - 2^{k_r}),$$

so

$$m \dotplus k = \mu \dotplus (k - 2^{k_r}) \leqq \mu + k - 2^{k_r} < k.$$

Hence the k indices $m \dotplus k$ satisfy $0 \leqq m \dotplus k < k$ and are distinct;

$$\sum_{n \in A} c_n \psi_n(t) = \sum_{n=0}^{k-1} c_n \psi_n(t) = s_k(t).$$

Thus we get

$$s_k(t)\psi_k(t) = g_{k_1}(t) + g_{k_2}(t) + \cdots + g_{k_\lambda}(t).$$

Put

$$p_n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_\lambda}, \qquad k_1 = k_1(n), \cdots, k_\lambda = k_\lambda(n);$$
$$\lambda = \lambda(n), \qquad 0 \leqq k_1 < k_2 < \cdots.$$

Then we would have, by Lemma 2,

$$\begin{aligned}
\int_0^1 \left(\sum_{n=1}^{\infty} | s_{n,p_n}(t) |^2 \right)^{r/2} dt
&= \int_0^1 \left(\sum_{n=1}^{\infty} | g_{n,k_1} + \cdots + g_{n,k_\lambda} |^2 \right)^{r/2} dt \\
&\leqq B_r \int_0^1 \left(\sum_{n=1}^{\infty} \sum_{\nu=1}^{\lambda(n)} | g_{n,k_\nu}(t) |^2 \right)^{r/2} dt \\
&\leqq B_r \int_0^1 \left(\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} | g_{n,k}(t) |^2 \right)^{r/2} dt \\
&= B_r \int_0^1 \left(\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \Delta_k^2(\psi_{p_n}(t) f_n(t)) \right)^{r/2} dt \\
&\leqq B_r D_r \int_0^1 \left(\sum_{n=1}^{\infty} | \psi_{p_n}(t) f_n(t) |^2 \right)^{r/2} dt \\
&= B_r D_r \int_0^1 \left(\sum_{n=1}^{\infty} | f_n(t) |^2 \right)^{r/2} dt,
\end{aligned}$$

which is the required inequality.

LEMMA 3. *If* $f(t) \in L^r$ $(r > 1)$, *then we have*

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{|s_n(t) - \sigma_n(t)|^2}{n} \right)^{r/2} dt$$

$$\leqq E_r \int_0^1 \left(\sum_{n=1}^{\infty} |s_{2^n}(t) - \sigma_{2^n}(t)|^2 \right)^{r/2} dt$$

$$\leqq F_r \int_0^1 \left(\sum_{n=0}^{\infty} |\Delta_n(t)|^2 \right)^{r/2} dt$$

$$\leqq G_r \int_0^1 \left(\sum_{n=1}^{\infty} \frac{|s_n(t) - \sigma_n(t)|^2}{n} \right)^{r/2} dt.$$

Except for the last inequality, the lemma is proved by Zygmund [4] for the case of Fourier series. He proved it by using the Fourier series analogue of Theorem 1. His method is also applicable for our case. The proof of the last inequality runs as follows. Since

$$s_n(t) = \sum_{\nu=0}^{n} c_\nu \psi_\nu(t),$$

$$\sigma_n(t) = \sum_{\nu=0}^{n} s_\nu(t)/(n+1) = \sum_{\nu=0}^{n} (1 - \nu/(n+1)) c_\nu \psi_\nu(t),$$

we have

$$s_n(t) = (r+1)\sigma_n(t) - n\sigma_{n-1}(t), \quad s_n(t) - \sigma_n(t) = n(\sigma_n(t) - \sigma_{n-1}(t))$$

and

$$s_n(t) - \sigma_n(t) = \{1c_1\psi_1(t) + 2c_2\psi_2(t) + \cdots + nc_n\psi_n(t)\}/(n+1).$$

Hence

$$\begin{aligned} |\sigma_{2^{n+1}}(t) - \sigma_{2^n}(t)| &\leqq \sum_{i=2^n+1}^{2^{n+1}} |\sigma_i(t) - \sigma_{i-1}(t)| \\ &\leqq \sum_{i=2^n+1}^{2^{n+1}} i(\sigma_i(t) - \sigma_{i-1}(t))^2 \sum_{i=2^n+1}^{2^{n+1}} 1/i \\ &\leqq 2 \sum_{i=2^n+1}^{2^{n+1}} i(\sigma_i(t) - \sigma_{i-1}(t))^2 \\ &\leqq 2 \sum_{i=2^n+1}^{2^{n+1}} |s_i(t) - \sigma_i(t)|^2/i. \end{aligned}$$

On the other hand

$$\int_0^1 \left(\sum_{k=1}^{\infty} | s_{2^k}(t) - \sigma_{2^k}(t) |^2 \right)^{r/2} dt$$

$$\leqq \int_0^1 \left\{ \sum_{k=1}^{\infty} \frac{1}{(2^k+1)^2} \left(\sum_{i=1}^{2^k} ic_i\psi_i(t) \right)^2 \right\}^{r/2} dt$$

$$\leqq 2 \int_0^1 \left\{ \sum_{k=1}^{\infty} \left(\sum_{i=1}^{2^k} ic_i\psi_i(t) \right)^2 \sum_{\nu=2^k}^{2^{k+1}-1} 1/\nu^3 \right\}^{r/2} dt$$

$$\leqq A_r \int_0^1 \left\{ \sum_{k=1}^{\infty} \sum_{\nu=2^k}^{2^{k+1}-1} \frac{1}{\nu^3} \left(\sum_{i=1}^{\nu} ic_i\psi_i(t) \right)^2 \right\}^{r/2} dt$$

$$\leqq B_r \int_0^1 \left(\sum_{r=1}^{\infty} \frac{| s_n(t) - \sigma_n(t) |^2}{n} \right)^{r/2} dt,$$

by Theorem 1. Since

$$| s_{2^{n+1}}(t) - s_{2^n}(t) | \leqq | s_{2^{n+1}}(t) - \sigma_{2^{n+1}}(t) | + | s_{2^n}(t) - \sigma_{2^n}(t) | + | \sigma_{2^{n+1}}(t) - \sigma_{2^n}(t) |,$$

we get the required inequality.

By Lemmas 2 and 3, Theorems 2 and 3 are now immediate.

LEMMA 4. *If $r>1$, then*

$$A_r \int_0^1 \left\{ \sum_{n=1}^{\infty} n(\sigma_n(t) - \sigma_{n-1}(t))^2 \right\}^{r/2} dt \leqq \int_0^1 | f(t) |^2 dt.$$

Since

$$s_n(t) = (n+1)\sigma_n(t) - n\sigma_{n-1}(t),$$

and

$$n(\sigma_n(t) - \sigma_{n-1}(t)) = s_n(t) - \sigma_n(t)$$

the lemma follows from Theorem 2.

LEMMA 5. *If $r>1$, then*

$$\int_0^1 \left(\sup_{0 \leqq n < \infty} | s_{2^n}(t) | \right)^r dt \leqq A_r \int_0^1 | f(t) |^r dt.$$

This is due to Paley [2].

With these preparations we will now prove Theorem 5. For $2^n < k < 2^{n+1}$, we have

$$(\sigma_k - \sigma_{2^n})^2 \leqq \left\{ \sum_{i=2^n+1}^{2^{n+1}} | \sigma_i(t) - \sigma_{i-1}(t) | \right\}^2$$

$$\leqq \sum_{i=2^n+1}^{2^{n+1}} i(\sigma_i(t) - \sigma_{i-1}(t))^2 \sum_{i=2^n+1}^{2^{n+1}} 1/i$$

$$= 2 \sum_{i=2^n+1}^{2^{n+1}} i(\sigma_i(t) - \sigma_{i-1}(t))^2.$$

Since

$$\sigma_k(t) = \sigma_{2^n}(t) + (\sigma_k(t) - \sigma_{2^n}(t)),$$

we have

$$\sup |\sigma_k(t)| \leqq \sup \sigma_{2^n}(t) + 2\sum_{i=1}^{\infty} i(\sigma_i(t) - \sigma_{i-1}(t))^2.$$

By Lemmas 4 and 5, we get Theorem 4.

Theorem 5 is easily deduced from Theorems 4 and 2.

Theorem 6 is proved by Zygmund's argument [5].

Bibliography

1. J. Marcinkiewicz, *Sur les multiplicateurs des séries de Fourier*, Studia Mathematica vol. 8 (1939) pp. 78–91.

2. R. E. A. C. Paley, *A remarkable series of orthogonal functions* (I), Proc. London Math. Soc. vol. 34 (1931) pp. 241–264.

3. ———, *A remarkable series of orthogonal functions* (II), Proc. London Math. Soc. vol. 34 (1931) pp. 265–279.

4. A. Zygmund, *On the convergence and summability of power series on the circle of convergence* (I), Fund. Math. vol. 30 (1938) pp. 170–196.

5. ———, *Proof of a theorem of Paley*, Proc. Cambridge Philos. Soc. vol. 34 (1938) pp. 125–133.

Mathematical Institute of Tôhoku University