ON A THEOREM OF GLEASON

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In a recent paper (*Square roots in locally euclidean groups*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 446–449), A. M. Gleason proved that, in a locally euclidean group G which has no small subgroups, there exist neighborhoods M and N of the neutral element e such that every element in M has a unique square root in N. The author clearly considered this result to be a step towards proving that, in such a group, some neighborhood of e is entirely filled by one-parameter subgroups of G. We shall establish here that he was justified in his expectation. As a matter of fact, our present proof requires only the local uniqueness of the square root, not its existence. This local uniqueness is established in the second part of the proof of Theorem 4 in Gleason's paper; and it may be observed that this part of the argument is independent of the assumption that the group be locally euclidean: it works equally well under the weaker assumption that the group be locally compact. We therefore have the following result:

LEMMA 1 (GLEASON). *Let G be a locally compact group in which there exists a neighborhood of the neutral element e which does not contain any subgroup $\neq \{e\}$ of G; then there is a neighbourhood V of e in G such that distinct elements of V have distinct squares.*

If we take V to be compact, then V is mapped topologically under the mapping $s \rightarrow s^2$.

Now we prove

THEOREM 1. *Let G be a locally euclidean group. Assume that there exists a neighborhood of the neutral element e of G which does not contain any subgroup $\neq \{e\}$ of G. Then there exists a neighborhood P of e which has the following property: if $t \in P$, there exists a uniquely determined continuous homomorphism $x \rightarrow s_t(x)$ of the additive group of real numbers into G such that $s_t(x) \in P$ for $-1 \leqq x \leqq 1$ and $s_t(1) = t$; $s_t(x)$ is a continuous function of the pair (t, x).*

It follows from Lemma 1 that there exists a neighborhood V of e which has the following properties: V is compact; $V = V^{-1}$; V is mapped topologically under the mapping $s \rightarrow s^2$; V does not contain any subgroup $\neq \{e\}$ of G; some open set containing V is homeo-

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morphic to an open solid sphere in a euclidean space. If n is any integer >0, we denote by Q_n the set of elements $u \in V$ such that $u^m \in V$ for $1 \leqq m \leqq 2^n$. If $u \in Q_n$, then we have also $u^m \in V$ for $-2^n \leqq m \leqq 2^n$. The set Q_n is obviously closed, and therefore compact. The mapping $s \to s^2$ being topological on V, the mapping $u \to u^{2^n}$ is a homeomorphism of Q_n with some compact subset P_n of V. If $t \in P_n$, we denote by $f_n(t)$ the element u of Q_n such that $u^{2^n} = t$; f_n is a homeomorphism of P_n with Q_n.

Let (t_n) be a sequence of elements of V which converges to an element t of V. Assume that $t_n \in P_n$ for every n. Then we shall prove that there exists a homomorphism $x \to s(x)$ of the group of real numbers into G such that $s(1) = t$ and $s(x) \in V$ for all $x \in [-1, +1]$. Let us select once and for all a nontrivial ultrafilter $\mathfrak{F}$ on the set of positive integers (that is, an ultrafilter which does not consist of all sets containing a certain integer). Let x be any real number in $[-1, +1]$, and let (m_n) be a sequence of integers such that $|m_n| \leqq 2^n$ for every n and $\lim_{\mathfrak{F}} 2^{-n} m_n = x$. Since V is compact, the limit $\lim_{\mathfrak{F}} (f_n(t_n))^{m_n}$ exists. We shall see that this limit depends only on x, not on the choice of the sequence (m_n). Let (m_n') be any sequence which satisfies the same conditions as (m_n); set $s = \lim_{\mathfrak{F}} (f_n(t_n))^{m_n}$ and $s' = \lim_{\mathfrak{F}} (f_n(t_n))^{m_n'}$. Denote by u the element $s^{-1}s'$, and by k an arbitrary integer. Then, clearly, $u^k = \lim_{\mathfrak{F}} (f_n(t_n))^{k(m_n' - m_n)}$; on the other hand, we have $\lim_{\mathfrak{F}} 2^{-n} k(m_n' - m_n) = 0$. It follows that there exists a set F in $\mathfrak{F}$ such that $|k(m_n' - m_n)| \leqq 2^n$ for all $n \in F$; this implies that $(f_n(t))^{k(m_n' - m_n)}$ is in V when $n \in F$, whence $u^k \in V$. Since V does not contain any subgroup $\neq \{e\}$ of G, we have $u = e$, whence $s' = s$. We denote by $s_t(x)$ the limit $\lim_{\mathfrak{F}} (f_n(t_n))^{m_n}$. If x, x', and $x + x'$ are in $[-1, +1]$, then it is easily seen that we can find sequences (m_n) and (m_n') such that $|m_n| \leqq 2^n$, $|m_n'| \leqq 2^n$, $|m_n + m_n'| \leqq 2^n$ and such that the sequences $(2^{-n} m_n)$ and $(2^{-n} m_n')$ converge (for instance in the ordinary sense) to x and x'; then the sequence $(2^{-n}(m_n + m_n'))$ converges to $x + x'$, and it follows immediately that $s_t(x + x') = s_t(x) s_t(x')$. We conclude from this that the mapping $x \to s_t(x)$ may be extended in one and only one way to a homomorphism of the additive group of real numbers (cf., for instance, C. Chevalley, *Theory of Lie groups*, Theorem 3, chap. 2, §8, p. 49). It is clear that $s_t(1) = t$.

It is obvious that $Q_{p+1} \subset Q_p$ for every $p > 0$. We shall prove that, given any neighborhood N of e, we have $Q_p \subset N$ as soon as p is large enough. We may assume N to be open; the sets $Q_p - (Q_p \cap N)$ are then compact and they form a monotone decreasing sequence. Were not Q_p contained in N from a certain p on, then there would exist a point s in the intersection of all sets $Q_p - (Q_p \cap N)$; we would have $s \neq e$, but $s^m \in V$ for every integer m, which is impossible. Using the same notation as above, let us prove that $s_t(x) \in Q_p$ whenever $|x| \leqq 2^{-p}$. If $|x| \leqq 2^{-p}$, we can find a sequence (m_n) such that $\lim (2^{-n}m_n) = x$ and $2^{-n}|m_n| \leqq 2^{-p}$ for every n. Since $|m_n| \leqq 2^{n-p}$, we have $(((f_n(t_n))^{m_n})^m \in V$ for all integers m such that $|m| \leqq 2^p$, and therefore $(f_n(t_n))^{m_n} \in Q_p$, whence, since Q_p is closed, $s_t(x) \in Q_p$.

Let P be the intersection of the sets P_n $(1 \leqq n < \infty)$. Then $s_t(x)$ is in particular defined whenever $t \in P$ (we take $t_n = t$ for every n). Since $s_t(x+x') = s_t(x)s_t(x')$ and $s_t(x) \in Q_p$ when $|x| \leqq 2^{-p}$, the mappings $x \rightarrow s_t(x)$ form a family of equicontinuous mappings of the additive group of real numbers into G. We have $s_t(1) = t$ and $s_t(x) \in V$ for $x \in [-1, +1]$. Now, let t be any point of P and σ a continuous homomorphism of the additive group of real numbers into G such that $\sigma(1) = t$ and $\sigma(x) \in V$ for $x \in [-1, +1]$. Then it is clear that $\sigma(2^{-n}) \in Q_n$ for every $n > 0$, whence $\sigma(2^{-n}) = f_n(t)$ and therefore $\sigma(2^{-n}m_n) = (f_n(t))^{m_n}$ for every integer m_n. Let x be in $[-1, +1]$, and let (m_n) be a sequence such that $\lim 2^{-n}m_n = x$, $|m_n| \leqq 2^n$; then, we have $\sigma(x) = \lim \sigma(2^{-n}m_n) = \lim (f_n(t_n))^{m_n} = \lim_{\mathfrak{F}} (f_n(t_n))^{m_n} = s_t(x)$. It follows immediately that $\sigma(x) = s_t(x)$ for every real number x. Now, let $\mathfrak{G}$ be any filter on P which converges to t. Then, for any fixed integer n, we have $\lim_{\mathfrak{G}} f_n(t') = f_n(t)$, whence $\lim_{\mathfrak{G}} s_{t'}(2^{-n}m) = s_t(2^{-n}m)$ for every integer m. This fact, together with the equicontinuity of the mappings s_t, shows that, when t' converges to t, the mapping $s_{t'}$ converges to s_t, uniformly on every compact subset of the additive group R of real numbers; this means that the mapping $(x, t) \rightarrow s_t(x)$ of $R \times P$ into G is continuous. Moreover, it is clear that, if $t \in P$, then $s_t(x) \in P$ for all x in $[-1, +1]$.

The assumption that G is locally euclidean has not been used as yet (the argument would go through in any locally compact group which has no small subgroups). We shall use this assumption to prove that P is a neighborhood of e. If u is an interior point of Q_n, then u^{2^n} is an interior point of P_n. For, let W be any neighborhood of u in Q_n; the mapping $u \rightarrow u^{2^n}$ induces a homeomorphism of W with a subset of V, and this subset is open in virtue of the theorem of invariance of the domain, which shows that u^{2^n} is interior to P_n. Now, it is clear that, for every n, Q_n is a neighborhood of e; it follows that P_n is a neighborhood of e. Since G is locally connected, in order to prove that P is a neighborhood of e, it will be sufficient to prove that it is impossible to have a sequence (t_n) of elements converging to e such that each t_n belongs to the boundary B_n of P_n. Now, if $t_n \in B_n$, one at least of the points $(f_n(t_n))^{m_n}$ $(1 \leqq m_n \leqq 2^n)$ is on the boundary of V. For, were these points all interior to V, then, obviously, some neighborhood of $f_n(t_n)$ would be in Q_n, and t_n would therefore be interior to P_n, as we have seen above. Let us then select a sequence (m_n) such that $1 \leqq m_n \leqq 2^n$ and that $(f_n(t_n))^{m_n}$ is on the boundary of V. The limit $x_0 = \lim_{\mathfrak{F}} 2^{-n} m_n$ exists because the interval $[-1, +1]$ is compact. We may now apply the result proved above: there exists a homomorphism $x \rightarrow s(x)$ of the group of real numbers into G such that $s(x_0) = \lim_{\mathfrak{F}} (f_n(t_n))^{m_n}$ is on the boundary of V, $s(1) = \lim t_n = e$ and $s(x) \in V$ for $x \in [-1, +1]$. But then the set of elements $s(x)$, for all real x, is a group $\neq \{e\}$ contained in V, and we arrive at a contradiction. Theorem 1 is thereby proved.

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