ON THE MINIMUM OF A CERTAIN INTEGRAL

A. SPITZBART

In this paper the following result will be proved.

Let $f(w)$ be an analytic function of w for $|w|<1$, continuous for $|w|\leqq 1$, and let the value $f'(\alpha)=1$ be prescribed at a point $w=\alpha$ within the unit circle. Among functions of this type, the minimum value of the integral $\int_C|f(w)|^p|dw|$, where $p\geqq 1$ and C is the unit circle $|w|=1$, is given by

$$\phi_1(|\alpha|, p) \text{ if } 1 \leqq p \leqq 1+|\alpha|, \qquad \phi_2(|\alpha|, p) \text{ if } p \geqq 1+|\alpha|,$$

where

$$\begin{aligned}\phi_1(|\alpha|, p) &= 2\pi(1-|\alpha|^2)^{p+1}[2(1+|\alpha|^2)]^{1-p}\\ &\quad\cdot[(p-1)|\alpha| + (|\alpha|^2-p^2+2p)^{1/2}]^{p-2}\\ &\quad\cdot[p+|\alpha|^2-|\alpha|(|\alpha|^2-p^2+2p)^{1/2}],\\ \phi_2(|\alpha|, p) &= 2\pi(1-|\alpha|^2)^{p+1}(p-1)^{2p-2}[(p-1)^2+|\alpha|^2]^{1-p}.\end{aligned}$$

These minima are attained.

As would be expected the two forms coincide if $p=1+|\alpha|$.

If $p=1$ the first form always applies and the minimum is $\phi_1(|\alpha|, 1)$. For $f(w)$ as in the statement of the theorem we have

$$\begin{aligned}\int_C |f(w)||dw| &\geqq \phi_1(|\alpha|, 1)\\ &= 2\pi(1-|\alpha|^2)^2[|\alpha|^2+(1+|\alpha|^2)^{1/2}]^{-1},\end{aligned}$$

a result which has been proved by Macintyre and Rogosinski.[1]

If $p\geqq 2$ the second form applies and, in particular, for $p=2$ the inequality becomes

$$\int_C |f(w)|^2|dw| \geqq \phi_2(|\alpha|, 2) = 2\pi(1-|\alpha|^2)^3(1+|\alpha|^2)^{-1}.$$

If $\alpha=0$ the second form applies so that with $f'(0)=1$ we have

$$\int_C |f(w)|^p|dw| \geqq 2\pi.$$

We proceed to the proof. By a particularization of a result of

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[1] The Edinburgh Mathematical Notes vol. 35 (1945) pp. 1–3.

Kakeya,[2] of the functions $F(z)$ which are analytic for $|z|<1$, continuous for $|z|\leqq 1$, and with the values $F(0)=A$, $F'(0)=D$ assigned, the one which minimizes the integral $\int_{C'}|F(z)|^p|dz|$, with $p\geqq 1$, C': $|z|=1$, is given by

$$(1)\qquad F_0(z) = A[pDz/(2A)+1]^{2/p} \qquad \text{if} \quad |pD| \leqq |2A|,$$

$$(2)\qquad F_0(z) = -A(1-\bar{b}z)^{2/p-1}(z-b)/b \qquad \text{if} \quad |pD| \geqq |2A|,$$

where

$$(3)\qquad \begin{aligned} b = &-|D|\{|D|-[|D|^2-4|A|^2(2/p-1)]^{1/2}\} \\ &\div [2\bar{A}D(2/p-1)]. \end{aligned}$$

We mention that the radicand appearing in b is non-negative, and $|b|\leqq 1$, for $|pD|\geqq|2A|$. If $b=0$, $F_0(z)$ in (2) is to be taken as Dz.

The values of the minimum integrals are easily obtained and are, for the two forms (1) and (2) respectively,

$$(4)\qquad \int_{C'} |F_0(z)|^p|dz| = 2\pi|A|^p[1+|pD/(2A)|^2],$$

$$(5)\qquad \int_{C'} |F_0(z)|^p|dz| = 2\pi|A|^p(1+|b|^2)|b|^{-p}.$$

If b vanishes (5) reduces to $2\pi|D|^p$.

Let us now make the transformation $w=(z+\alpha)/(1+\bar{\alpha}z)$, $z=(w-\alpha)/(1-\bar{\alpha}w)$. In (4) and (5) the left members become

$$(1-|\alpha|^2)\int_C |F_0[(w-\alpha)/(1-\bar{\alpha}w)]\cdot(1-\bar{\alpha}w)^{-2/p}|^p|dw|.$$

Let us set $f(w)=(1-\bar{\alpha}w)^{-2/p}F_0[(w-\alpha)/(1-\bar{\alpha}w)]$, and write $f(\alpha)=A'$, $f'(\alpha)=1$, which gives the relations

$$(6)\quad A = A'(1-|\alpha|^2)^{2/p},\ D=(1-|\alpha|^2)^{2/p}[p(1-|\alpha|^2)-2\bar{\alpha}A']/p.$$

The method of proof is to minimize $\int_C|f(w)|^p|dw|$ for each of the two forms with respect to A', and compare the values thus obtained.

The case $p=2$. We consider first the case $p=2$, for which the forms (1) and (2) coincide and become $Dz+A$. We have

$$(7)\qquad \begin{aligned} \int_C |f(w)|^2|dw| &= 2\pi(1-|\alpha|^2)^{-1}[|A|^2+|D|^2] \\ &= 2\pi(1-|\alpha|^2)^{-1}[|A|^2+|(1-|\alpha|^2)^2-\bar{\alpha}A|^2]. \end{aligned}$$

[2] S. Kakeya, *General mean modulus of analytic functions*, Proceedings of the Physico-Mathematical Society of Japan (3) vol. 3 (1921) pp. 48–58.

If $\alpha=0$ the minimum occurs for $A=0$ and is $2\pi=\phi_2(0, 2)$. If $\alpha\neq 0$ then for any modulus of A the minimum of (7) occurs when A has the same amplitude as α, that is, if $A=k\alpha$ for some $k>0$, and (7) becomes

$$2\pi(1-|\alpha|^2)^{-1}[|\alpha|^2(1+|\alpha|^2)k^2 \\ -2|\alpha|^2(1-|\alpha|^2)^2k+(1-|\alpha|^2)^4].$$

The derivative with respect to k vanishes for $k=(1-|\alpha|^2)^2(1+|\alpha|^2)^{-1}$, which yields as the minimum the value $2\pi(1-|\alpha|^2)^3(1+|\alpha|^2)^{-1}=\phi_2(|\alpha|, 2)$.

Henceforth we exclude the value $p=2$.

The first form of $f(w)$. For $f(w)$ corresponding to $F_0(z)$ of the first form we have, with (4) and (6),

$$\int_C |f(w)|^p|dw| \\ = 2\pi(1-|\alpha|^2)|A'|^p[1+|p(1-|\alpha|^2)/(2A')-\bar{\alpha}|^2], \tag{8}$$

and for our problem this is to be minimized with respect to A'. The condition $|pD|\leqq|2A|$, which shall define the term *admissible* for the first form, becomes $|p(1-|\alpha|^2)/(2A')-\bar{\alpha}|\leqq 1$, which excludes for the first form the possibility that $A'=0$. If $\alpha=0$ the minimum of (8) subject to the condition on A' occurs for $|A'|=p/2$, and is $\phi_3(0, p)$, where

$$\phi_3(|\alpha|, p) = 2\pi\cdot 2^{1-p}p^p(1-|\alpha|^2)(1-|\alpha|)^p. \tag{9}$$

If $\alpha\neq 0$, for any given modulus of A' the minimum of (8) occurs when A' is a positive multiple of α, that is, if $A'=k'\alpha$ for some $k'>0$. Let us set $a=p(1-|\alpha|^2)/(2|\alpha|^2)$; the right member of (8) becomes, apart from a constant factor,

$$k'^p(1+|\alpha|^2)-2a|\alpha|^2k'^{p-1}+a^2|\alpha|^2k'^{p-2}, \tag{10}$$

and the condition on A' becomes $k'\geqq k_1'=p(1-|\alpha|)/(2|\alpha|)$.

If $p>1+(1+|\alpha|^2)^{1/2}$ the derivative of (10) with respect to k' vanishes for no positive value of k'.

If $p\leqq 1+(1+|\alpha|^2)^{1/2}$ the derivative of (10) vanishes for

$$k'=k_0'=(1-|\alpha|^2)[(p-1)|\alpha|+(|\alpha|^2-p^2+2p)^{1/2}] \\ \div [2|\alpha|(1+|\alpha|^2)]. \tag{11}$$

We are concerned here with the relation of magnitude of k_1' and k_0'. If $1\leqq p\leqq 1+|\alpha|$ only the positive root in k_0' gives a positive k_0', and we may show that $k_0'\geqq k_1'$, so that the minimum of (10) occurs for $k'=k_0'$ and the minimum of (8) is $\phi_1(|\alpha|, p)$, which is a relative minimum.

Now suppose $p>1+|\alpha|$, so that $k_0'<k_1'$, and k_0' is not an admissible value of k'. Let us write

$$A' = k'|\alpha|e^{i\theta}, \qquad \alpha = |\alpha|e^{i\theta_0}.$$

Then (8) is a function of k' and θ. For any value $k'>0$ the minimum of (8) occurs for $\theta=\theta_0$, the maximum occurs for $\theta=\theta_0+\pi$. If we let $t=e^{i(\theta-\theta_0)}+e^{-i(\theta-\theta_0)}$, the value in (8) becomes, apart from a constant factor,

$$\begin{aligned}4|\alpha|^2(1+|\alpha|^2)k'^p - 2p|\alpha|^2t(1-|\alpha|^2)k'^{p-1}\\ + p^2(1-|\alpha|^2)^2k'^{p-2},\end{aligned} \tag{12}$$

and the derivative of (12) with respect to k' is

$$\begin{aligned}pk'^{p-3}[4|\alpha|^2(1+|\alpha|^2)k'^2 - 2|\alpha|^2(p-1)t(1-|\alpha|^2)k'\\ + p(p-2)(1-|\alpha|^2)],\end{aligned} \tag{13}$$

with the values of t between -2 and 2.

If $1+|\alpha|<p<2$, for each value of t there is one positive zero of (13). These zeros give the minima of (12) with respect to k' for the different values of θ. The relative minimum of these minima occurs for $t=2$, or $\theta=\theta_0$, and is not admissible; hence the admissible minimum, if any, occurs where

$$|p(1-|\alpha|^2)/(2A')-\bar{\alpha}| = 1. \tag{14}$$

If none of these minima is admissible, the admissible minimum of (12) certainly occurs where (14) holds. In this event (8) reduces to $2\pi \cdot 2(1-|\alpha|^2)|A'|^p$ for which the minimum subject to (14) occurs for $A'=k_1'\alpha$, and the minimum of (8) is $\phi_3(|\alpha|, p)$.

If $p>2$ we have the following situation. For $t\leqq 0$ there is no positive zero of (13), and (12) increases with respect to k'. For $t>0$ there are no positive zeros if

$$p > 1 + 2(1+|\alpha|^2)^{1/2}[4(1+|\alpha|^2) - t^2|\alpha|^2]^{-1/2},$$

and two positive zeros if this inequality is reversed. The larger of these zeros gives the relative minimum for a fixed $t>0$. Again the minimum of these minima occurs for $\theta=\theta_0$ and is not admissible. The admissible minimum again occurs where (14) holds, and gives A' and $\phi_3(|\alpha|, p)$ as above.

For $f(w)$ corresponding to the first form of $F_0(z)$ the result is therefore that the minimum is $\phi_1(|\alpha|, p)$ if $1\leqq p\leqq 1+|\alpha|$, and is $\phi_3(|\alpha|, p)$

if $1+|\alpha| < p < 2$, or $p > 2$.

The second form of $f(w)$. With (5) and (6) we have for the second form of $f(w)$.

$$\int_C |f(w)|^p |dw| = 2\pi |A|^p (1 + |b|^2)[(1 - |\alpha|^2)|b|^p]^{-1}, \tag{15}$$

with b, A, D as at the start of the proof. We shall mean by *admissible* for the second form that the condition $|pD| \geqq |2A|$, $|b| \leqq 1$ is satisfied.

We consider first the case $1 \leqq p < 2$. Let us set

$$R = |D| - [|D|^2 - 4|A|^2(2/p - 1)]^{1/2} \qquad (0 \leqq R \leqq |D|).$$

Then

$$\begin{aligned} |A|^2 &= (2R|D| - R^2)[2(2/p - 1)]^{-1}, \\ |b| &= R[2|A|(2/p - 1)]^{-1} \end{aligned} \tag{16}$$

and (15) becomes

$$\begin{aligned} &2\pi \cdot 2^{1-p}[(1 - |\alpha|^2)(2 - p)]^{-1}(2|D| - R)^{p-1} \\ &\quad \cdot [|D|(2 - p) - R(1 - p)]. \end{aligned} \tag{17}$$

Although $R=0$ is initially exceptional, (17) is valid also for $R=0$.

If $\alpha=0$ and $p=1$, (17) has the value 2π. If $\alpha=0$ and $1<p<2$, (17) is valid and is a function of $|A|$ alone, since $D=1$. Its derivative with respect to $|A|$ vanishes only for $A=0$, in which case the value of (17) is again 2π. Hence if $\alpha=0$ and $1 \leqq p < 2$, the minimum of (17) is 2π.

If $\alpha \neq 0$ let us again set

$$A = k|\alpha| e^{i\theta}, \qquad \alpha = |\alpha| e^{i\theta_0}.$$

The expression in (17) is a function of k and θ. If $k=0$ the value of (17) is constant. For fixed $k>0$ the derivative with respect to θ of the part of (17) involving k and θ becomes

$$p(2 - p)(2|D| - R)^{p-1}\partial|D|/\partial\theta. \tag{18}$$

Now

$$\begin{aligned} |D|^2 = p^{-2}[&p^2(1 - |\alpha|^2)^{4/p+2} \\ &- 2p(1 - |\alpha|^2)^{2/p+1}|\alpha|^2 k(e^{i(\theta-\theta_0)} + e^{-i(\theta-\theta_0)}) + 4|\alpha|^4 k^2] \end{aligned}$$

so that

$$2|D| \cdot \partial|D|/\partial\theta = -2ip^{-1}(1 - |\alpha|^2)^{2/p+1}|\alpha|^2 k(e^{i(\theta-\theta_0)} - e^{-i(\theta-\theta_0)}).$$

Hence (18) vanishes only if $\theta=\theta_0$ or $\theta=\theta_0+\pi$ (since $R=2|D|$ is not admissible) so that for a fixed $k>0$ the minimum of (17) occurs for $\theta=\theta_0$, the maximum for $\theta=\theta_0+\pi$.

Let us now minimize (17) with respect to k for $\theta=\theta_0$. The derivative with respect to k of the part of (17) involving k and θ becomes

$$\begin{aligned}(19)\qquad & p(2|D|-R)^{p-2}\{(1-p)(R-|D|)\partial R/\partial k \\ & \qquad + [(2p-3)R+2(2-p)|D|]\partial|D|/\partial k\}.\end{aligned}$$

We have $\partial R/\partial k=[R\partial|D|/\partial k-4|\alpha|^2k(2/p-1)](R-|D|)^{-1}$. The case $R=|D|$ is admissible only if $p=1$, in which case $|pD|=|2A|$; this is considered in the discussion of the second form for $1\leqq p<1+|\alpha|$. With $R\neq|D|$, (19) becomes

$$p(2|D|-R)^{p-2}(2-p)[(2|D|-R)\partial|D|/\partial k-4|\alpha|^2(1-p)p^{-1}k],$$

which vanishes for $R=2|D|-4(1-p)|\alpha|^2k[p\partial|D|/\partial k]^{-1}$. With $\theta=\theta_0$ the value of D is real and is $D=p^{-1}[p(1-|\alpha|^2)^{2/p+1}-2k|\alpha|^2]$. Only if $D>0$ will k be admissible, so that $\partial|D|/\partial k=-2|\alpha|^2/p$. Thus $R=2|D|+2(1-p)k$ and we have

$$2|D|+2(1-p)k=|D|-[|D|^2-4|A|^2(2/p-1)]^{1/2}.$$

With $|A|=k|\alpha|$ the only possibly valid solution of this equation is

$$(20)\qquad k=(p-1)(1-|\alpha|^2)^{2/p+1}[(p-1)^2+|\alpha|^2]^{-1}.$$

With this value of k the value of $|b|$ in (16) is computed as $|b|=|\alpha|/(p-1)$, and the value of k in (20) is thus admissible if and only if $|\alpha|\leqq p-1$, in which case this value of k and $\theta=\theta_0$ actually furnish the minimum, a relative minimum whose value is computed as $\phi_2(|\alpha|, p)$ as given in the statement of the theorem.

It has been shown that the minimum of $f(w)$ of the second form is $\phi_2(|\alpha|, p)$ if $1<1+|\alpha|\leqq p<2$, and is 2π if $\alpha=0$. We may consistently define $\phi_2(0, p)=2\pi$. Hence the minimum of $f(w)$ of the second form is $\phi_2(|\alpha|, p)$ if $1+|\alpha|\leqq p<2$.

Let us consider the second form of $f(w)$ for $p>2$. Here we set

$$R=[|D|^2+4|A|^2(1-2/p)]^{1/2}-|D|.$$

The value of (15) now becomes

$$\begin{aligned}(21)\qquad & 2\pi\cdot 2^{1-p}[(1-|\alpha|^2)(p-2)]^{-1}(2|D|+R)^{p-1} \\ & \qquad\qquad \cdot[(p-2)|D|+(p-1)R].\end{aligned}$$

If $\alpha=0$ we have $D=1$ so that again (21) is a function of $|A|$ alone, its derivative with respect to $|A|$ vanishes only for $A=0$, and the minimum value is 2π. If $\alpha \neq 0$ we again set $A = k|\alpha|e^{i\theta}$, $\alpha = |\alpha|e^{i\theta_0}$. As in the case $p<2$, for fixed k the minimum of (21) occurs for $\theta = \theta_0$, and with $\theta = \theta_0$ the minimizing value of k is (20). With these values of θ and k we have $|b| = |\alpha|/(p-1)$, so that $|b|<1$ for $p>2$. The values of k and θ are admissible and the minimum for the second form with $p>2$ is $\phi_2(|\alpha|, p)$.

A combination of the results now permits us to state that the minimum of $f(w)$ of the second form is $\phi_2(|\alpha|, p)$ whenever $1+|\alpha| \leqq p$.

We turn to a discussion of (17) when $1 \leqq p < 1+|\alpha|$, in which case we must minimize (17) subject to the condition $|pD| \geqq |2A|$. It has been shown that the relative minimum for fixed k occurs for $\theta = \theta_0$. For a given θ the admissible minimum of (17) with respect to k occurs for some value of k. For that value of k the admissible minimum with respect to θ occurs either for $\theta = \theta_0$ or where $|pD| = |2A|$. Among the values of (17) for $\theta = \theta_0$ the admissible minimum when $1 \leqq p < 1+|\alpha|$ again occurs where $|pD| = |2A|$. Hence in any event the admissible minimum of (17) occurs where $|pD| = |2A|$, in which case (17) reduces to $2\pi \cdot 2(1-|\alpha|^2)^{-1}|A|^p$, the minimum of $|A|$ occurs for $A = p\alpha(1-|\alpha|)(1-|\alpha|^2)^{2/p}(2|\alpha|)^{-1}$, and the minimum value is $\phi_3(|\alpha|, p)$, which appears in (9).

The results thus far are the following, with $p=2$ again included. If $1 \leqq p \leqq 1+|\alpha|$ the minimum is $\phi_1(|\alpha|, p)$ for the first form, and $\phi_3(|\alpha|, p)$ for the second form. If $p > 1+|\alpha|$ the minimum is $\phi_3(|\alpha|, p)$ for the first form and $\phi_2(|\alpha|, p)$ for the second form. We must now compare the two minima for each range of values of p.

I. We wish to show that $\phi_1(|\alpha|, p) < \phi_3(|\alpha|, p)$ if $1 \leqq p < 1+|\alpha|$. Let

$$x = p + |\alpha|^2 - |\alpha|(|\alpha|^2 - p^2 + 2p)^{1/2},$$
$$y = (p-1)|\alpha| + (|\alpha|^2 - p^2 + 2p)^{1/2}.$$

Then $\phi_1(|\alpha|, p) < \phi_3(|\alpha|, p)$ if

$$(1+|\alpha|)(1+|\alpha|^2)^{-1} < py^{2/p-1}[(1+|\alpha|^2)x]^{-1/p},$$

which is in turn valid if

$$\log [(1+|\alpha|)(1+|\alpha|^2)^{-1}] < \log \{py^{2/p-1}[(1+|\alpha|^2)x]^{-1/p}\}.$$

The two members are equal for $p = 1+|\alpha|$; the inequality is therefore valid for $1 \leqq p < 1+|\alpha|$ if the derivatives with respect to p satisfy the reversed inequality, which becomes

$$0 > \log [(1+|\alpha|^2)xy^{-2}]$$

since $xy-(p-2)x\,dy/dp-y\,dx/dp=0$. The last inequality is valid if

$$(1+|\alpha|^2)xy^{-2}<1.$$

Now, $2(1+|\alpha|^2)x=x^2+y^2$ so that the last inequality is valid if $x^2-y^2<0$, which can easily be proved if $1\leqq p<1+|\alpha|$. The desired inequality is thus proved.

II. We wish to show here that $\phi_2(|\alpha|, p)<\phi_3(|\alpha|, p)$ if $p>1+|\alpha|$. If $\alpha=0$ it is easy to see that the inequality holds. If $\alpha\neq 0$ let $p-1=q$. Then $\phi_2(|\alpha|, p)<\phi_3(|\alpha|, p)$ if

$$[(q^2+|\alpha|^2)/(2q^2)]^q>[(1+|\alpha|)/(q+1)]^{q+1},$$

which is valid if their logarithms are in the same relation:

$$q[\log(q^2+|\alpha|^2)-\log(2q^2)]>(q+1)[\log(1+|\alpha|)-\log(q+1)].$$

The two members are equal if $q=|\alpha|$; hence the inequality is valid for $q>|\alpha|$ if the derivatives are in the same relation, the resulting inequality becoming

$$\log(q^2+|\alpha|^2)-\log(2q^2)-2|\alpha|^2(q^2+|\alpha|^2)^{-1}+1$$
$$-\log(1+|\alpha|)+\log(q+1)>0.$$

The two members are again equal if $q=|\alpha|$; hence this inequality is valid for $q>|\alpha|$ if the derivative of the left member is positive, which statement may be expressed as

$$P(q)=q^5+4|\alpha|^2q^3+2|\alpha|^2q^2-|\alpha|^4q-2|\alpha|^4>0.$$

The equation $P(q)=0$ has one variation in sign, hence by Descartes' rule of signs at most one positive root. But $P(0)=-2|\alpha|^4<0$, and $P(|\alpha|)=4|\alpha|^5>0$, so that there is a positive root, it lies between $q=0$ and $q=|\alpha|$, and for $q>|\alpha|$ the last inequality above is valid, and the proof is complete that $\phi_2(|\alpha|, p)<\phi_3(|\alpha|, p)$ if $q>|\alpha|$, or if $p>1+|\alpha|$.

The proof of the theorem is now complete.

In conclusion we mention that the minimizing function is unique except when $\alpha=0$, $p=1$. If $p<1+|\alpha|$, the minimizing function does not vanish for $|w|\leqq 1$. If $p>1+|\alpha|$, the minimizing function has a simple zero within the unit circle.

UNIVERSITY OF WISCONSIN IN MILWAUKEE