

AN EXTENSION OF THE NOMOGRAMIC INSTRUMENT

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The usual nomogram involves three variables and has the straight line for its instrument. We may say, therefore, that its instrument is a member of the family of straight lines

$$A_1x_1 + A_2x_2 + A_3 = 0.$$

More generally, a hyperplane in n -dimensional space is usually necessary for $n+1$ variables. By permitting more complicated instruments it may be possible to reduce the number of dimensions in which the instrument operates. The theorem to be given does precisely this by selecting the instrument from a certain family of hypersurfaces.

DEFINITION. If f is a function of n variables s_1, \dots, s_n and can be expressed by an n by n determinant (V_{ij}) where V_{ij} is a function of s_i only, then f will be called a nomogramic disjunctive function, or simply an N-function.

DEFINITION. Let R_i^k be a function of the single variable s_i for $k=1, \dots, m$ and for $i=1, \dots, n$. Let g_j be a function of m variables u_1, \dots, u_m for $j=1, \dots, n$. We shall say that the triple (m, R_i^k, g_j) belongs to f if f is an N-function with

$$V_{ij} = g_j(R_i^1, \dots, R_i^m) \quad \text{for } i, j = 1, \dots, n.$$

(k is a superscript.)

LEMMA. Every N-function has at least one triple.

PROOF. Let f be an N-function. Then take

$$m = n, \quad g_j(u_1, \dots, u_n) = u_j, \quad R_i^k = V_{ik}.$$

THEOREM. Let (m, R_i^k, g_j) belong to f . Let f be defined on D , a subset of n -dimensional euclidean space E^n . Let T_1, \dots, T_m be m arbitrary functions of a single variable which have inverses on the domains considered. Let C_1, \dots, C_n be n curves in E^m given parametrically by

$$C_i: \quad x_k = T_k^{-1}(R_i^k(s_i)), \quad k = 1, \dots, m; i = 1, \dots, n.$$

Finally, let F be the family of hypersurfaces indicated by

$$A_1g_1(T_1(x_1), \dots, T_m(x_m)) + \dots + A_n g_n(T_1(x_1), \dots, T_m(x_m)) = 0$$

where the A 's are in a field embedding the ranges of the g 's. Then for

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each (s_1, \dots, s_n) which annuls f there is a set of n points $(x_1, \dots, x_m)_i$ on C_i for $i=1, \dots, n$ which lie on a single member of F . The coordinates of these n points are obtained from the parametric expressions for the curves.

Conversely, if a member of F intersects C_1, \dots, C_n in the n points $(x_1, \dots, x_m)_i$ for $i=1, \dots, n$, respectively, which arise from s_1, \dots, s_n , respectively, in constructing the curves, then f is annulled by (s_1, \dots, s_n) .

PROOF. Let $f(\bar{s})=0$ where $\bar{s}=(\bar{s}_1, \dots, \bar{s}_n)$. Then the determinant expression for f vanishes for \bar{s} . Each \bar{s}_i determines a single point on C_i namely, $(\bar{x}_1, \dots, \bar{x}_m)_i$. Put these n values one at a time into the general expression for F and consider A_1, \dots, A_n as the unknowns to be solved for. Then the resulting set of n equations has a coefficient determinant which vanishes as it is precisely $f(\bar{s})$. Hence, there exists a nontrivial set of values for A_1, \dots, A_n which determines a member of F , and this member vanishes on all n points $(\bar{x}_1, \dots, \bar{x}_m)_i$, $i=1, \dots, n$.

For the converse, let h , a member of F , intersect C_i in $(\bar{x}_1, \dots, \bar{x}_m)_i$ which gives rise to \bar{s}_i , $i=1, \dots, n$. Let the point $(\bar{s}_1, \dots, \bar{s}_n)$ belong to D . Now each $(\bar{x}_1, \dots, \bar{x}_m)_i$ annuls h . Consider A_1, \dots, A_n as unknowns in these n vanishing expressions, then we have a solution to a set of n homogeneous linear equations and this solution will not be trivial if we avoid the zero member of F . Hence, the determinant of the coefficients of the A 's, which is precisely $f(\bar{s})$, must vanish. This completes the proof of the theorem.

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