CERTAIN PROPERTIES OF FUNCTIONS HARMONIC WITHIN A SPHERE

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1. **Introduction.**[1] Let S be the sphere of radius a about the origin O of the rectangular coordinate system (x, y, z). Let V be the interior of S. Let $U=U(x, y, z)=U(r, \theta, \phi)$ be the function, harmonic in V, given by the Poisson integral of $u(\theta, \phi)$ over S,

$$U(P) = \frac{a^2 - r^2}{4\pi a}\int\int_S \frac{u(Q)}{[PQ]^3}\,dS_Q \qquad (r = [OP]), \tag{1.1}$$

where $u(\theta, \phi) \in L$ on S. When the relation (1.1) holds between U and u, we write $U=p(u)$.

The following Theorem I extends to three dimensions results obtained by Douglas for two dimensions (see [2, pp. 307–311]).[2]

THEOREM I. *Let $u(P)\in L$ on S and U be the function, harmonic in V, such that $U=p(u)$. Let P_n (cos θ) and P_n^m(cos θ) be respectively the Legendre polynomial of order n, and the associated Legendre function of the first kind. Denote the Laplace expansion of u on S by*

$$u(\theta, \phi) \sim \sum_{n=0}^{\infty}\Bigg[A_nP_n(\cos\theta) + \sum_{m=1}^{n}(A_{n,m}\cos m\phi + B_{n,m}\sin m\phi)P_n^m(\cos\theta)\Bigg].$$

Then, if any one of the three numbers

$$A[U] = \int\int\int_V |\nabla U(P)|^2 dV_P,$$

$$B[u] = \frac{1}{4\pi}\int\int_S dS_M \int\int_S \frac{[u(M) - u(N)]^2}{[MN]^3}\,dS_N,$$

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[1] The results of this paper appear in Part I of the author's thesis prepared under the direction of Professor J. J. Gergen and submitted in April, 1949, to the Graduate School of Arts and Sciences of Duke University in partial fulfillment of the requirements for the degree of Doctor of Philosophy. Theorem I was first stated by the author in Bull. Amer. Math. Soc. Abstract 54-3-139 presented to the American Mathematical Society.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

$$C[u] = \sum_{n=1}^{\infty} \frac{2\pi na}{2n+1}\left[2A_n^2 + \sum_{m=1}^{n} \frac{(n+m)!}{(n-m)!}(A_{n,m}^2 + B_{n,m}^2)\right],$$

is finite, the other two are finite and all three are equal.

It has been shown by Bray and Evans [**1**, pp. 179–180] that, if $U(P)$ is harmonic in V, and if $A[U] < +\infty$, then there exists $u(\theta, \phi) \in L$ on S such that $U = p(u)$. The following theorem is an immediate corollary of Theorem I and this result of Bray and Evans.

THEOREM II. *Let $U(P)$ be harmonic in V. Then $A[U]$ is finite if, and only if, there exists a function $u(\theta, \phi)$ belonging to L on S such that $U = p(u)$ and $B[u]$ is finite.*

The final result of this paper is Theorem III which establishes an inequality similar to one developed by J. Hadamard [**3**, pp. 135–138] for the analogous two-dimensional case.

THEOREM III. (a) *Let $u(P) \in L$ on S. Let $W(P) \in C'$ in V, and suppose that $W \to u$, $r \to a^-$, on almost all radii of S. Then, $A[U] \leqq A[W]$ where $U = p(u)$.*

(b) *Further, the inequality holds, unless $A[U] = +\infty$, or U is identical with W.*

2. **Lemma for Theorem I.** The proof of Theorem I depends on the following lemma.

LEMMA 1. *For M on S, and P in V, put*

$$F(P, M) = \frac{a^2 - r^2}{[PM]^3} \qquad (r = [OP]).$$

Let $G(P, M)$ denote the radial derivative at P of $F(P, M)$, regarded as a function of P:

$$(2.1) \quad G(P, M) = \frac{\partial}{\partial r}F(P, M) = \frac{-2r}{[PM]^3} - \frac{3(a^2 - r^2)}{[PM]^4} \cos(\angle OPM).$$

Denote by S_r the sphere of radius r about o. Put

$$(2.2) \qquad H(r, M, N) = \iint_{S_r} F(P, M)G(P, N)dS_P.$$

Then there exists a constant K, such that

$$(2.3) \qquad [MN]^3 \left| H(r, M, N) \right| \leqq K$$

for M, N on S, $0 < r < a$. Further, if M, N are on S, $M \neq N$, then

$$\lim_{r\to a^-} H(r, M, N) = -\frac{8\pi a^2}{[MN]^3}. \tag{2.4}$$

PROOF. For P in V, $[OP]=r$, and M, N on S, we have

$$F(P, M) = \frac{1}{a}\sum_{k=0}^{\infty}(2k+1)P_k(\cos\angle MOP)\left(\frac{r}{a}\right)^k, \tag{2.5}$$

$$G(P, N) = \frac{1}{a^2}\sum_{k=1}^{\infty}k(2k+1)P_k(\cos\angle NOP)\left(\frac{r}{a}\right)^{k-1}. \tag{2.6}$$

For r held fast, $0<r<a$, and M, N fixed on S, these series are uniformly convergent for P on S_r. Making use of the formulas,

$$\iint_{S_r} P_j(\cos\angle MOP)P_k(\cos\angle NOP)dS_P$$

$$= \begin{cases} 0 & (j \neq k), \\ \dfrac{4\pi r^2}{2k+1} P_k(\cos\angle MON) & (j = k), \end{cases}$$

we obtain for $0<r<a$,

$$H(r, M, N) = \frac{4\pi r}{a^2}\sum_{k=1}^{\infty}k(2k+1)P_k(\cos\angle MON)\left(\frac{r}{a}\right)^{2k}. \tag{2.7}$$

Let $\rho=r^2/a$; let Q be the point of intersection of $[OM]$ and S. From (2.7) and (2.6), we get

$$H(r, M, N) = 4\pi\frac{\rho}{a}(a\rho)^{1/2}G(Q, N). \tag{2.8}$$

Hence, by (2.1),

$$H(r, M, N) = -4\pi\frac{\rho}{a}(a\rho)^{1/2}\left[\frac{2}{[QN]^3}+\frac{3(a^2-\rho^2)}{[QN]^4}\cos\angle OQN\right] \tag{2.9}$$

from which (2.4) follows. Further, from the equality

$$[QN]^2 = (a-\rho)^2 + \frac{\rho}{a}[MN]^2$$

we obtain, for $0<r<a$, $[MN] \leqq (a/\rho)^{1/2}[QN]$, $a-\rho \leqq [QN]$. These inequalities with (2.9) give

$$\left| H(r, M, N)\cdot[MN]^3\right| \leqq 4\pi\frac{\rho}{a}(a\rho)^{1/2}\frac{8a}{[QN]^3}\cdot[MN]^3 \leqq 32\pi a^2.$$

3. Proof of Theorem I.

(A) We assume first that

$$B[u] < +\infty. \tag{3.1}$$

We prove, under this hypothesis, that

$$A[U] = B[u]. \tag{3.2}$$

Under (3.1), for almost all points M on S, the integral

$$\iint_S \frac{[u(M) - u(N)]^2}{[MN]^3} dS_N \tag{3.3}$$

exists as a finite number. Selecting any point M for which (3.3) is finite, we get

$$\iint_S u^2(N)dS_N \leqq 16a^3 \iint_S \frac{[u(M) - u(N)]^2}{[MN]^3} dS_N + 8\pi a^2 u^2(M) < +\infty.$$

Thus $u \in L^2$ on S.

Now let $w_1(P)$, $w_2(P) \in L$ on S. Let $W_1(P) = p(w_1)$, $W_2(P) = p(w_2)$. By Green's theorem, if V_r is the interior of the sphere S_r, $0 < r < a$,

$$\iiint_{V_r} \nabla W_1 \cdot \nabla W_2 dV = \iint_{S_r} W_1(P) \frac{\partial}{\partial r} W_2(P) dS_P. \tag{3.4}$$

Thus

$$\iiint_{V_r} \nabla W_1 \cdot \nabla W_2 dV = \frac{1}{16\pi^2 a^2} \iint_{S_r} dS_P \iint_S F(P, M) w_1(M) dS_M \iint_S G(P, N) w_2(N) dS_N,$$

which by (2.2) and Fubini's theorem can be written

$$\iiint_{V_r} \nabla W_1 \cdot \nabla W_2 dV = \frac{1}{16\pi^2 a^2} \iint_S dS_M \iint_S w_1(M) w_2(N) H(r, M, N) dS_N. \tag{3.5}$$

Applying (3.5) in the three cases:

(a) $w_1 = u^2$, $w_2 = 1$, (b) $w_1 = 1$, $w_2 = u^2$, (c) $w_1 = u$, $w_2 = u$, we obtain

$$0 = \frac{1}{32\pi^2 a^2} \iint_S dS_M \iint_S u^2(M) H(r, M, N) dS_N,$$

$$0 = \frac{1}{32\pi^2 a^2} \iint_S dS_M \iint_S u^2(N) H(r, M, N) dS_N,$$

$$\iiint_{V_r} |\nabla U|^2 dV = \frac{1}{32\pi^2 a^2} \iint_S dS_M \iint_S 2u(M)u(N) H(r, M, N) dS_N.$$

From these three equalities, we get

$$(3.6)\quad A_r[U] = -\frac{1}{32\pi^2 a^2} \iint_S dS_M \iint_S [u(M) - u(N)]^2 H(r, M, N) dS_N,$$

where $A_r[U] = \iiint_{V_r} |\nabla U|^2 dV$.

From (3.6), (3.1), Lemma 1, and Lebesgue's theorem, we get

$$(3.7)\qquad \lim_{r \to a^-} A_r[U] = \frac{1}{4\pi} \iint_S dS_M \iint_S \frac{[u(M) - u(N)]^2}{[MN]^3} dS_N.$$

Since $\lim_{r \to a^-} A_r[U] = A[U]$, the truth of (3.2) follows.

(B) We now assume

$$(3.8)\qquad A[U] < +\infty.$$

We prove that (3.8) implies (3.1). The function $U(P)$ is harmonic in V. Hence, if r is held fast, $0 < r < a$, it follows that $U(M') - U(N')$ is bounded by a constant multiple of $[M'N']$ for M', N' on S_r. Thus

$$(3.9)\qquad \frac{1}{4\pi} \iint_{S_r} dS_{M'} \iint_{S_r} \frac{[U(M') - U(N')]^2}{[M'N']^3} dS_{N'} < +\infty.$$

Accordingly, by what we proved in (A),

$$\iiint_{V_r} |\nabla U(P)|^2 dV_P = \frac{1}{4\pi} \iint_{S_r} dS_{M'} \iint_{S_r} \frac{[U(M') - U(N')]^2}{[M'N']^3} dS_{N'},$$

which can be written

$$(3.10)\quad A_r[U] = \frac{1}{4\pi} \frac{r}{a} \iint_S \iint_S \frac{[U(M') - U(N')]^2}{[MN]^3} dS_M dS_N,$$

where M', N' are the intersections of $[OM]$, $[ON]$, with S_r. We have

$U(M') \to u(M)$, $r \to a^-$ for almost all points M of S. Thus, by Fatou's lemma, we get

$$(3.11)\quad \begin{aligned} B[u] &= \frac{1}{4\pi}\iint_S dS_M \iint_S \liminf_{r\to a^-} \left\{\frac{r}{a}\,\frac{[U(M') - U(N')]^2}{[MN]^3}\right\} dS_N \\ &\leqq \liminf_{r\to a^-} A_r[U] = A[U] < +\infty, \end{aligned}$$

which is (3.1).

(C) To complete the proof of Theorem I it suffices to show that

$$(3.12)\qquad A[U] = C[u]$$

for $A[U]$, $C[u]$ finite or $+\infty$.

We have, for $P(r, \theta, \phi)$ in V,

$$(3.13)\quad \begin{aligned} U(P) = \sum_{n=0}^{\infty}\Bigg[&A_n P_n(\cos\theta) \\ &+ \sum_{m=1}^{n} (A_{n,m}\cos m\phi + B_{n,m}\sin m\phi) P_n^m(\cos\theta)\Bigg]\left(\frac{r}{a}\right)^n. \end{aligned}$$

The series (3.13) for $U(P)$ and the series for $\partial U/\partial r$ obtained by termwise differentiation of (3.13) are uniformly convergent series of spherical harmonics for r held fast, $0<r<a$. By substituting these series for U and $\partial U/\partial r$ in (3.4), by using the orthogonality properties of spherical harmonics, and by using the formulas for computing the coefficients of the Laplace series, we get

$$(3.14)\quad \begin{aligned} A_r[U] = \sum_{n=1}^{\infty} \frac{2an\pi}{2n+1}\Bigg[2A_n^2 + \sum_{m=1}^{n} \frac{(n+m)!}{(n-m)!}(A_{n,m}^2 \\ + B_{n,m}^2)\Bigg]\left(\frac{r}{a}\right)^{2n+1}. \end{aligned}$$

Since $\lim_{r\to a^-} A_r[U] = A[U]$, the proof of (3.12) for $A[U]$, $C[u]$ finite or $+\infty$ follows from (3.14) and Abel's lemma. This completes the proof of Theorem I.

4. **Lemma for Theorem III.** We base the proof of Theorem III on the following lemma.

LEMMA 2. *Let $W \in C''$ in a domain containing $S + V$. Then*

$$(4.1)\quad B[W] = \frac{1}{4\pi}\iint_S dS_M \iint_S \frac{[W(M) - W(N)]^2}{[MN]^3}\, dS_N \leqq A[W].$$

PROOF. Let $U=p(W)$. We can write $U=U_1+U_2$, where U_1 $[U_2]$ is a potential of simple [double] distribution of class C'' on S (see [4, p. 241]). It follows that U_1 and U_2 are continuously differentiable on $S+V$ (see [4, pp. 160–172]). Hence, in particular, U has bounded first partial derivatives in V. We have, for $0<b<a$,

$$\begin{aligned} A_b[W] &= A_b[W-U]+A_b[U]+2\iiint_{V_b}\nabla U\cdot\nabla(W-U)dV \\ &= A_b[W-U]+A_b[U]+2\iint_{S_b}\frac{\partial U}{\partial r}(W-U)dS, \end{aligned} \tag{4.2}$$

by Green's theorem. Hence

$$\begin{aligned} A_b[U] &\leqq A_b[W]+2\left|\iint_{S_b}\frac{\partial U}{\partial r}(W-U)dS\right| \\ &\leqq A[W]+2\left|\iint_{S_b}\frac{\partial U}{\partial r}(W-U)dS\right|. \end{aligned} \tag{4.3}$$

We know that $W-U$ is continuous on $V+S$, and vanishes on S, and U has bounded first partial derivatives in V. Hence,

$$\iint_{S_b}\frac{\partial U}{\partial r}(W-U)dS\to 0 \qquad (b\to a^-).$$

Further, $A_b[U]\to A[U]$, $b\to a^-$. It follows from (4.3) that

$$A[U]\leqq A[W]. \tag{4.4}$$

By Theorem I, we have

$$A[U]=B[W], \tag{4.5}$$

since $U=p(W)$. The lemma follows from (4.4) and (4.5).

5. **Proof of Theorem III.** (a) Suppose $0<b<a$. For $n>3^{1/2}/(a-b)$, put

$$W_n(x,y,z)=n^3\int_x^{x+1/n}dx\int_y^{y+1/n}dy\int_z^{z+1/n}W(x,y,z)dz.$$

Then W_n is of class C'' in a domain containing S_b+V_b. Hence, by Lemma 2,

$$B_b[W_n]=\frac{1}{4\pi}\iint_{S_b}dS_M\iint_{S_b}\frac{[W_n(M)-W_n(N)]^2}{[MN]^3}dS_N\leqq A_b[W_n]. \tag{5.1}$$

Now $W_n(M) \to W(M)$, $n \to \infty$, on S_b; and the first partial derivatives of W_n tend uniformly to those of W in V_b. Hence,

$$B_b[W] \leqq \liminf_{n \to \infty} B_b[W_n], \tag{5.2}$$

by Fatou's lemma, and

$$A_b[W] = \liminf_{n \to \infty} A_b[W_n]. \tag{5.3}$$

Thus,

$$B_b[W] \leqq \liminf_{n \to \infty} B_b[W_n] \leqq \liminf_{n \to \infty} A_b[W_n] \leqq A_b[W], \tag{5.4}$$

by (5.1), (5.2), and (5.3). It follows that

$$B_b[W] \leqq A[W], \tag{5.5}$$

since $A_b[W] \leqq A[W]$.

Now $B_b[W]$ may be written

$$B_b[W] = \frac{1}{4\pi} \frac{b}{a} \iint_S dS_M \iint_S \frac{[W(M_b) - W(N_b)]^2}{[MN]^3} dS_N, \tag{5.6}$$

where M_b and N_b are the intersections of $[OM]$ and $[ON]$ with S_b. But $W(M_b) \to u(M)$, $b \to a^-$ for almost all points M of S. Thus, by Fatou's lemma, and (5.5), we get

$$B[u] \leqq A[W]. \tag{5.7}$$

By Theorem I, (5.7) implies $A[U] \leqq A[W]$ which completes the proof of part (a).

(b) Assume $A[U] < +\infty$, and $U \not\equiv W$. Then we have $0 < A[(U-W)/2]$. Put $W_1 = (U+W)/2$. Then W_1 satisfies the condition imposed on W in (a). Hence, by (a), $A[U] \leqq A[W_1]$. Thus,

$$\begin{aligned} A[U] &\leqq 2A[W_1] - A[U] \\ &< 2A[W_1] + 2A[(U-W)/2] - A[U] \\ &= 2A[(U+W)/2] + 2A[(U-W)/2] - A[U] \\ &= A[U] + A[W] - A[U] \\ &= A[W], \end{aligned} \tag{5.8}$$

or $A[U] < A[W]$. This completes the proof of Theorem III.

BIBLIOGRAPHY

1. H. E. Bray and G. C. Evans, *A class of functions harmonic within the sphere*, Amer. J. Math. vol. 49 (1927) pp. 153–180.

2. Jesse Douglas, *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. vol. 33 (1931) pp. 263–321.

3. J. Hadamard, *Sur le principe de Dirichlet*, Bull. Soc. Math. France vol. 34 (1906) pp. 135–138.

4. Oliver Dimon Kellogg, *Foundations of potential theory*, Berlin, Springer, 1929.

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NON-MEASURABLE SETS AND THE EQUATION $f(x+y)=f(x)+f(y)$

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1. A set of S real numbers which has inner measure $m_*(S)$ different from its outer measure $m^*(S)$ is non-measurable. An extreme form, which we shall call saturated non-measurability, occurs when $m_*(S)=0$ but $m^*(SM)=m(M)$ for every measurable set M, $m(M)$ denoting the measure of M. This is equivalent to: both S and its complement have zero inner measure.

More generally, if a fixed set B of positive measure is under consideration, a subset S of B will be called s-non-mble. if both S and its complement relative to B have zero inner measure. This implies $m_*(S)=0$, $m^*(S)=m(B)$ but is implied by these conditions only if $m(B)$ is finite.

Our object, in part, is to show that if B is either the set of all real numbers or any half-open finite interval, then for every infinite cardinal $k\leqq C$ (the power of the continuum), B can be partitioned into k disjoint subsets which are s-non-mble. and are mutually congruent under translation (modulo the length of B in the case that B is a finite interval). Sierpinski and Lusin[1] have partitioned B into continuum many disjoint s-non-mble. subsets but they are not constructed to be congruent under translation. Other well known constructions do partition B into a countable infinity of mutually congruent non-measurable subsets, but the subsets are not constructed to be saturated non-measurable.[2]

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[1] C. R. Acad. Sci. Paris vol. 165 (1917) pp. 422–424.

[2] See Hahn and Rosenthal, *Set functions*, University of New Mexico Press, 1948, pp. 102–104. The construction of §8.3.3 on p. 102 (as will be shown below) does give an s-non-mble. set but this is not proved there.