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ON THE CONTINUITY OF PARAMETRIC LINEAR OPERATIONS[1]

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The proofs of the theorems asserting strong continuity for semi-groups of linear operations in Banach spaces usually involve measurability and integrability of Banach-space-valued functions [2, pp. 183–184].[2] A theorem of this type in which the assumptions and the proof are purely topological is given below.

Let G be both a topological space and an additive group, and let H be a subset of G. For each h in H let $D(h)$ be all points g in H satisfying these two conditions: (1) $h-g\in H$, (2) for each open set N_g about g there is an open set N_h about h such that $h-g+(H\cap N_g) \supset H\cap N_h$. Letting X be a complex linear normed space, a set $\Gamma=[\gamma]$ of bounded complex linear functionals on X is *a total set for* E, E a subset of X, if $\|x\|=\sup\ [|\gamma(x)|,\ \gamma\in\Gamma]$ holds for every x in the smallest linear subspace containing E. A function T_h on H to the

Presented to the Society, April 21, 1951; received by the editors May 29, 1950.

[1] This paper was written under Contract N7-onr-434, Task Order III, Navy Department (The Office of Naval Research).

[2] Numbers in brackets refer to the references at the end of the paper.

space $B(X)$ of bounded linear operations on X to X is *additive* if $T_{h+k}=T_h(T_k)$ whenever h, k, and $h+k$ are in H. Finally, we define a function $\phi(a)$ on a topological space A to a metric space B to be *residually separably-valued* if the set $\phi(A')\equiv[\phi(a)\mid a\in A']$ is separable in B for some set A' residual in A, that is, for some A' having its complement of the first category.

THEOREM. *Let H be a subset of G and let T_h be additive on H to $B(X)$. Suppose that* (i) *$D(h)$ is second category for each $h\in H$, and that* (ii) *for each x_0 in X the function $\phi(h)\equiv T_h(x_0)$ on H to X is residually separably-valued and there is a total set $\Gamma(x_0)$ for $\phi(H)$ such that $\gamma(\phi(h))$ is continuous on H to complex numbers for each $\gamma\in\Gamma(x_0)$. Then $\phi(h)$ is continuous on H to X for each $x_0\in X$.*

Fixing x_0 in X, the idea of the proof is to show, using (ii), that the set C of points of continuity of ϕ is residual, and then to apply (i) to show that $C=H$.

The proof that C is residual is due to Alexiewicz and Orlicz [**1**, pp. 107–108 and 114–115] in a slightly more restricted case. Their arguments can be adapted here as follows. We first observe that if S is any closed sphere in X with center y in $\phi(H)$ and radius r, then $\phi^{-1}(S)$ is closed in H; for since $\Gamma(x_0)$ is a total set for $\phi(H)$ and $\gamma(\phi(h))$ is continuous for $\gamma\in\Gamma(x_0)$, clearly $\phi^{-1}(S)=H[h\mid \|\phi(h)-y\|\leqq r]=H[h\mid \sup_\gamma |\gamma(\phi(h))-\gamma(y)|\leqq r]=\bigcap_\gamma H[h\mid |\gamma(\phi(h))-\gamma(y)|\leqq r]$, a closed set. To establish that C is residual it is enough to prove this: *if ϕ is any function on a topological space H to a metric space X and ϕ is residually separably-valued and $\phi^{-1}(S)$ is closed whenever S is a closed sphere with center in $\phi(H)$, then ϕ has its points of continuity forming a residual set C.* Let R be a residual set such that $\phi(R)$ is separable in X. Since $\phi(R)$ is separable, there exist countably many closed spheres $\{S_n\}$ with centers in $\phi(R)$ such that for any open set V in X we have $V\cap\phi(R)=\bigcup[S_n\cap\phi(R)\mid S_n\subset V]=\bigcup_k(S_{n_k}\cap\phi(R))$ where n_k ranges over all n such that $S_n\subset V$. Let R' be the complement of R in H and set $P_V=R'\cap\phi^{-1}(V)$. Then $\phi^{-1}(V)=P_V\cup[R\cap\phi^{-1}(V)]\subset P_V\cup\phi^{-1}(V\cap\phi(R))\subset P_V\cup(\bigcup_k\phi^{-1}(S_{n_k}))\subset\phi^{-1}(V)$, since each $S_{n_k}\subset V$; hence $\phi^{-1}(V)=P_V\cup(\bigcup_k F_{n_k})$, where $F_{n_k}=\phi^{-1}(S_{n_k})$, and in particular $\bigcup_k(F^0_{n_k})\subset\phi^{-1}(V)^0$, where E^0 denotes the interior of any set E. Now set $Q=R'\cup(\bigcup_n(F_n-F^0_n))$; since R is residual and each F_n is closed, Q is a first category set. Moreover, clearly $\phi^{-1}(V)\subset P_V\cup(\bigcup_k(F_{n_k}-F^0_{n_k}))\cup(\bigcup_k F^0_{n_k})\subset Q\cup\phi^{-1}(V)^0$. Now in H consider any point g not in Q and any open V containing $\phi(g)$; since $g\in\phi^{-1}(V)\subset Q\cup\phi^{-1}(V)^0$ and $g\notin Q$, obviously $g\in\phi^{-1}(V)^0$ and hence ϕ is continuous at g. Thus C is residual.

Let h be fixed in H. Since $D(h)$ is second category there exists a point g in $D(h)$ at which ϕ is continuous. Set $\rho = \|T_{h-g}\|$ and let $\epsilon > 0$ be given. Having ϕ continuous at g, there is an open set N_g about g such that $\|\phi(k) - \phi(g)\| < \epsilon/\rho$ whenever $k \in H \cap N_g$. Then, since $g \in D(H)$, there is an open set N_h about h such that $h - g + (H \cap N_g) \supset H \cap N_h$. Consider any h' in $H \cap N_h$. Writing $h' = h - g + k$ where $k \in H \cap N_g$, we have $\|\phi(h') - \phi(h)\| = \|\phi(h-g+k) - \phi(h-g+g)\| = \|T_{h-g}(T_k(x_0)) - T_{h-g}(T_g(x_0))\| \leqq \rho \|\phi(k) - \phi(g)\| < \epsilon$, showing that ϕ is continuous at h and ending the proof.

It is easy to verify that assumption (i) in the theorem is satisfied when (i′) the group sum $h+k$ in G is continuous in k and for each h in H the set $H \cap (-H+h)$ is in the interior of H and is second category. Statement (i′) in turn is implied by this; (i″) $h+k$ is continuous in each variable in G, G is second category, H and $-H$ are open, and $H \subset H+H$. Condition (i″) holds, for example, when G is a second category linear topological space and H is an open convex set having the zero element as a limit point. Assumption (ii) of the theorem follows if H contains a countable dense subset and $\gamma(T_h(x_0))$ is continuous on H to complex numbers for each x_0 in X and each bounded linear functional γ on X. In particular, Theorem 9.2.2 of [2] now results.

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