NOTE ON THE TRANSCENDENCE OF CERTAIN SERIES

WILLIAM J. LEVEQUE

1. **Introduction.** Let k and m be integers satisfying either of the sets of conditions, $k=0$ or 1, $m \geqq 0$, or $k \neq 0$, 1, and $m \geqq -1$. Then the series

$$\sum_{n=0}^{\infty} \frac{(n+k)^{n+m} x^n}{n!}$$

defines a function f_{km} in the range $|x| < e^{-1}$. It is the object of this note to show that $f_{km}(x)$ is transcendental for every algebraic $x \neq 0$ in this interval. The proof involves no new transcendence investigations, but depends directly on Lindemann's theorem that e^x is transcendental for algebraic $x \neq 0$. The connection between the exponential function and f_{km} is established by using the Lagrange expansion in powers of x

$$G(z) = G(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!} [D_t^{n-1} G'(t) \phi^n(t)]_{t=0} \tag{1}$$

of any function G (regular in a neighborhood of $z=0$) of the quantity z defined by the equation $x = z/\phi(z)$, where $\phi(z)$ is regular in a neighborhood of $z=0$ and $\phi(0) \neq 0$ (see Pólya-Szegö, *Analysis*, vol. 1, p. 125).

The symbol 0^0 is to be interpreted as 1, throughout. For brevity put $f_{kk} = f_k$.

2. **The case** $k=0$.

LEMMA 1. *The function y defined by the equation $y^y = e^{-x}$, and such that* $\lim_{x \to 0} y(x) = y(0) = 1$, *is identical with f_{-1} in the range* $|x| < e^{-1}$.

For it follows from $y^y = e^{-x}$ that $x = z/e^z$, where $z = -\log y$. Taking $\phi(z) = e^z$, $G(z) = e^{-z} = y$ in (1) gives

$$\begin{aligned} y(x) &= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} [D_t^{n-1}(-e^{(n-1)t})]_{t=0} \\ &= 1 - \sum_{n=1}^{\infty} \frac{(n-1)^{n-1} x^n}{n!} \\ &= -f_{-1}(x). \end{aligned}$$

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LEMMA 2. *$f_0(x)$ is transcendental for each algebraic x with $0<|x|<e^{-1}$.*

By Lemma 1, $y=-f_{-1}$ is a solution of the equation $\log y=-x/y$, so that not both of $x, f_{-1}(x)$ can be algebraic unless $x=0$. But $y\log y=-x$ implies that $y'=y/(x-y)$, so that also not both of $y'(x)=-f_0(x)$ and $x\neq 0$ can be algebraic.

LEMMA 3. *f_{0m} is a polynomial in f_0 with rational integral coefficients.*

Let $y^y=e^{-x}$, and put $y'=z=y/(x-y)$. Then it can be verified that $z'=z^2(1+z)/x$, or

$$f_0'(x) = -\frac{f_0^2(x)(1-f_0(x))}{x},$$

so that $xD_xf_0=P_1(f_0)$, where P_1 is a polynomial with integral coefficients. It follows from this that $(xD_x)^mf_0=P_m(f_0)$, where P_m is again a polynomial with integral coefficients. Finally, it is clear from the definition of f_{km} that $(xD_x)^mf_0=f_{0m}$.

THEOREM 1. *If $m\geqq 0, f_{0m}(x)$ is transcendental for every algebraic x for which $0<|x|<e^{-1}$.*

This follows immediately from Lemmas 2 and 3.

It might mentioned that f_0 is a solution of the equation $(1-1/y)e^{-(1-1/y)}=x$.

3. **The case** $k\neq 0$. Putting $y=z$, $G(y)=e^{ky}$, $\phi(y)=e^y$ in (1) gives the following statement.

LEMMA 4. *If y is that solution of the equation $ye^{-y}=x$ which is continuous at $x=0$, then for $k\neq 0$, $e^{ky}=kf_{k,-1}(x)$ for $|x|<e^{-1}$.*

LEMMA 5. *For $k\neq 1, f_{k0}(x)$ is transcendental for each algebraic x for which $0<|x|<e^{-1}$.*

Since $f'_{k-1,-1}=f_{k0}$, it follows from Lemma 4 that

$$e^{(k-1)y}y' = f_{k0},$$

and since $y'=e^y/(1-y)$,

$$f_{k0}(x) = \frac{e^{ky}}{1-y} = \frac{(y/x)^k}{1-y}.$$

By its definition $y(x)$ is transcendental for algebraic $x\neq 0$, so that also $f_{k0}(x)$ must be transcendental.

LEMMA 6. *If $m \geqq 0$, $k \neq 0, 1$, then*

$$f_{km}(x) = \frac{P_{km}(y)}{x^k},$$

where P_{km} is a nonconstant rational function with integral coefficients and y is the y of Lemma 4.

From the definition of f_{km}, it is seen that

$$(xD_x)^m(x^k f_{k0}) = x^k f_{km}.$$

On the other hand, using the conventional operator notation, we have

$$D_x = D_x y \cdot D_y = \frac{e^y}{1-y} D_y,$$

so that

$$xD_x = \frac{y}{1-y} D_y.$$

Hence

$$f_{km} = \frac{1}{x^k}\left(\frac{y}{1-y} D_y\right)^m \frac{y^k}{1-y};$$

the expression on the right contains a factor y^k in the numerator, while the denominator is of the form $x^k(1-y)^s$, and the proof is complete.

LEMMA 7. *For every m,*

$$xf_{1m}(x) = f_{0m}(x) - \begin{cases} 0 & \text{if } m \neq 0, \\ 1 & \text{if } m = 0, \end{cases}$$

and so for $m \geqq 0$, by Theorem 1, $f_{1m}(x)$ is transcendental for each algebraic x, $0 < |x| < e^{-1}$.

This follows immediately from the definition of f_{km}.

THEOREM 2. *If $k \neq 0$, 1 and $m \geqq -1$, or if $k = 1$ and $m \geqq 0$, the quantity $f_{km}(x)$ is transcendental for each algebraic x with $0 < |x| < e^{-1}$.*

UNIVERSITY OF MICHIGAN