7. ———, *Converse theorems of summability for Dirichlet's series*, Trans. Amer. Math. Soc. vol. 39 (1936) pp. 117–130.

8. D. V. Widder, *The Laplace transform*, Princeton Mathematical Series, no. 6, 1946.

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TYPICALLY-REAL FUNCTIONS WITH ASSIGNED ZEROS

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1. Introduction. A function $f(z)$

$$f(z) = \sum_{n=0}^{\infty} b_n z^n \tag{1.1}$$

is said to be typically-real of order p, if in (1.1) the coefficients b_n are all real and if either (I) $f(z)$ is regular in $|z| \leqq 1$ and $\Im f(e^{i\theta})$ changes sign $2p$ times as $z = e^{i\theta}$ traverses the boundary of the unit circle, or (II) $f(z)$ is regular in $|z| < 1$ and if there is a $\rho < 1$ such that for each r in $\rho < r < 1$, $\Im f(re^{i\theta})$ changes sign $2p$ times as $z = re^{i\theta}$ traverses the circle $|z| = r$. This set of functions is denoted by $T(p)$.

The name typically-real was first suggested by Rogosinski [6][1] who studied these functions in the case $p = 1$. The more general set of functions $T(p)$ was first introduced by Robertson [5; 4], and in a recent paper by Robertson and Goodman [3] the sharp upper bound for $|b_n|$ in terms of $|b_1|, \cdots, |b_p|$ was obtained, namely for $n = p+1, p+2, \cdots$,

$$|b_n| \leqq \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)} |b_k|. \tag{1.2}$$

We shall see in what follows that the size of $|b_n|$ is also governed by the locations of the zeros of $f(z)$ for functions of the set $T(p)$. More precisely we shall prove the following theorem.

THEOREM 1. *Let*

$$f(z) = z^q + \sum_{n=q+1}^{\infty} b_n z^n \tag{1.3}$$

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[1] Numbers in brackets refer to the references at the end of the paper.

be a function of the set $T(p)$. Suppose that in addition to the qth order zero at $z=0$, the function $f(z)$ has exactly s zeros,[2] $\beta_1, \beta_2, \cdots, \beta_s$, *such that $0<|\beta_j|<1, j=1, 2, \cdots, s$. Finally let the non-negative integer t be defined by*

$$q + s + t = p \geqq 1 \tag{1.4}$$

and let $m=[(t+1)/2]$. Then

$$|b_n| \leqq B_n, \qquad n = q+1, q+2, \cdots, \tag{1.5}$$

where B_n is defined by

$$\begin{aligned} F(z) &= \frac{z^q}{(1-z)^{2q+2s}}\left(\frac{1+z}{1-z}\right)^{2m} \prod_{j=1}^{s}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|) \\ &= z^q + \sum_{n=q+1}^{\infty} B_n z^n. \end{aligned} \tag{1.6}$$

When t is odd or when $t=0$, $F(z)\in T(p)$ and the inequality (1.5) is sharp.

Theorem 1 is to some extent a generalization of a recent result [2, Theorem 2], where an expression similar to (1.6) occurs. However in [2] the coefficients could be complex, so that neither theorem includes the other.

2. **The case** $q=s=0$. We first prove Theorem 1 in the simple case that $f(z)$ has no zeros in $|z|<1$. For convenience we use the notation $g(z)\ll G(z)$ to mean that if

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad G(z) = \sum_{n=0}^{\infty} A_n z^n \tag{2.1}$$

then

$$|a_n| \leqq A_n, \qquad n = 0, 1, 2, \cdots, \tag{2.2}$$

and under these conditions we shall say that $G(z)$ is a majorant of $g(z)$.

Now suppose that $g(z)=1+b_1z+b_2z^2+\cdots$ belongs to the set $T(t)$ and has no zeros in $|z|<1$. We are to prove that

$$g(z) \ll \left(\frac{1+z}{1-z}\right)^{2m}. \tag{2.3}$$

[2] Here and in the rest of the paper a kth order zero appears k times in the list $\beta_1, \beta_2, \cdots, \beta_s$, and s is the number of zeros β_j counted in accordance with their multiplicities.

Since $g(z)$ is free of zeros, there is a function $h(z)=(g(z))^{1/2m}$ such that

$$h(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n \tag{2.4}$$

is regular and has no zeros in $|z|<1$, and the α_n are all real. Consequently $h(r)>0$ for $-1<r<1$. We shall see that for $|z|<1$, $\Re h(z) \geqq 0$. Let us suppose that there is a z_1 inside the unit circle, for which $\Re h(z_1)<0$. If $z_2=\bar{z}_1$, then $\Re h(z_2)<0$ and further since z_1 is not real $z_2 \neq z_1$. If we define $\phi=\arg h(z)$ so that for real z, $\phi=0$, and for $|z|<1$, ϕ is a continuous function, it is clear that with a proper choice of notation we have $\phi_1=\arg h(z_1)<-\pi/2$ and $\phi_2=\arg h(z_2)>\pi/2$. Further, if not for $h(z)$ then for $h(-z)$, $-\pi<\theta_1=\arg z_1 \leqq 0$ and $\pi>\theta_2=\arg z_2>0$. Now consider $g(z)=(h(z))^{2m}$ on Γ the arc of the circle $z=|z_1|e^{i\theta}$, $\theta_1 \leqq \theta \leqq \theta_2$. Since $\arg g(z)=2m \arg h(z)$, it is clear that $\arg g(z)$ varies continuously from $2m\phi_1<-m\pi$ to $2m\phi_2>m\pi$ and hence $\Im g(z)$ changes sign at least $2m+1$ times on Γ. But $g(z)$ has no zeros, so that on the full circle $z=|z_1|e^{i\theta}$, $-\pi+\epsilon<\theta\leqq\pi+\epsilon$, $\Delta \arg g(z)=0$. Therefore on the full circle $\Im g(z)$ must change sign at least $4m+2$ times. This is a contradiction if $g(z)\in T(t)$ where $t\leqq 2m$. Hence if $m\geqq t/2$, $\Re h(z)\geqq 0$. By Carathéodory's Theorem, $h(z) \ll (1+z)/(1-z)$ and hence (2.3) is proved. If m and t are integers the restriction $t\leqq 2m$ is satisfied if $m=[(t+1)/2]$, and this is the least integer m which can be used.

3. **The general case.** It will be sufficient to prove (1.5) for functions $f_p(z)$ regular in $|z|\leqq 1$. For if $f(z)$ given by (1.3) is regular in $|z|<1$ and belongs to the set $T(p)$, then for each r, $\rho<r<1$,

$$f_p(z) = \frac{f(rz)}{r^q} = z^q + \sum_{n=q+1}^{\infty} b_n^{(p)} z^n \tag{3.1}$$

is also an element of $T(p)$ and is regular in $|z|\leqq 1$. If $r>\max\{|\beta_1|, \cdots, |\beta_s|\}$, the zeros of $f_p(z)$ will occur at $z=0$ (if $q>0$) and at $z=\beta_j/r$, $j=1, 2, \cdots, s$. Let us denote by $B_n^{(p)}$ the coefficients in the power series (1.6) when $|\beta_j|$ is replaced by $|\beta_j/r|$, $j=1, 2, \cdots, s$. Now $b_n^{(p)}$ and $B_n^{(p)}$ are continuous functions of r and $b_n^{(p)}\to b_n$, $B_n^{(p)}\to B_n$ as $r\to 1$. Therefore if there were a function $f(z)$ belonging to $T(p)$ for which $|b_n|>B_n$, by taking r sufficiently close to 1, a function $f_p(z)$ could be constructed, regular in $|z|\leqq 1$, and satisfying the conditions of Theorem 1, and for this function $|b_n^{(p)}| > B_n^{(p)}$.

We complete the proof of (1.5) by induction on q and s. We have already shown that the inequality holds when $q=s=0$. Consider the three auxiliary functions,

$$k_1(z) = \frac{1 - 2z\cos\nu + z^2}{z}, \tag{3.2}$$

$$k_2(z) = \frac{1 - 2z\cos\nu + z^2}{(1 + z/\beta)(1 + z\beta)}, \tag{3.3}$$

$$k_3(z) = \frac{(1 - 2z\cos\nu + z^2)(1 - 2z\cos\mu + z^2)}{(1 + z/\beta)(1 + z\beta)(1 + z/\bar{\beta})(1 + z\bar{\beta})}, \tag{3.4}$$

where ν and μ are real. In (3.3) β is real and $0<\beta^2<1$. In (3.4) β is not real and $0<|\beta|<1$.

On the boundary of the unit circle we have for these auxiliary functions

$$k_1(e^{i\theta}) = 2(\cos\theta - \cos\nu), \tag{3.5}$$

$$k_2(e^{i\theta}) = \frac{2(\cos\theta - \cos\nu)}{2\cos\theta + \beta + \beta^{-1}}, \tag{3.6}$$

$$k_3(e^{i\theta}) = (\cos\theta - \cos\nu)(\cos\theta - \cos\mu)/D_3(\theta, \beta) \tag{3.7}$$

where

$$D_3(\theta, \beta) = \cos^2\theta + (\eta + \bar{\eta})\cos\theta + \eta\bar{\eta}, \quad 2\eta = \beta + \beta^{-1}. \tag{3.8}$$

It is important to notice that in (3.6) and (3.7) the denominators are never zero for real θ, and hence have constant sign. This assertion is obvious for (3.6) since $0<\beta^2<1$. For (3.7), we need to remark only that $D_3(\theta, \beta)$ as a quadratic in $\cos\theta$ has the two complex roots $\cos\theta = -\eta, -\bar{\eta}$.

Now let $f_p(z)$ satisfy the conditions of Theorem 1, and suppose $q>0$, and $p>1$. Then $f_{p-1}(z)$ defined by $f_{p-1}(z) = k_1(z)f_p(z)$ is regular in $|z| \leqq 1$. Since

$$\Im f_{p-1}(e^{i\theta}) = 2(\cos\theta - \cos\nu)\Im f_p(e^{i\theta}), \tag{3.9}$$

it is possible to select ν in $k_1(z)$ so that[3] $f_{p-1}(z) \in T(p-1)$ as indicated by the subscript. By the induction hypothesis

$$f_{p-1}(z) \ll \frac{z^{q-1}}{(1-z)^{2q+2s-2}}\left(\frac{1+z}{1-z}\right)^{2m} \prod_{j=1}^{s}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|), \tag{3.10}$$

[3] Some details are omitted here. The technique is identical with that used in [3], where a more complete account may be found.

and since $k_1^{-1}(z) \ll z/(1-z)^2$, the bound (1.5) follows from (3.10).

Next suppose that $s>0$, $p>1$ and that $f_p(z)$ has a real zero β_s, $0<\beta_s^2<1$. Set $\beta=-\beta_s$ in $k_2(z)$ and define $f_{p-1}(z)$ by $f_{p-1}(z)=k_2(z)f_p(z)$. Just as before $f_{p-1}(z)$ is regular in $|z|\leqq 1$ and for this function

$$(3.11) \qquad \Im f_{p-1}(e^{i\theta}) = \frac{2(\cos\theta - \cos\nu)}{2\cos\theta - \beta_s - \beta_s^{-1}} \Im f_p(e^{i\theta}).$$

Again it is possible to select ν so that $f_{p-1}(z) \in T(p-1)$ as indicated by the subscript. By the induction hypothesis

$$(3.12) \quad f_{p-1}(z) \ll \frac{z^q}{(1-z)^{2q+2s-2}}\left(\frac{1+z}{1-z}\right)^{2m} \prod_{j=1}^{s-1}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|).$$

But $k_2^{-1}(z) \ll (1+z/|\beta_s|)(1+z|\beta_s|)/(1-z)^2$ and this together with (3.12) again gives the bound (1.5) for $f_p(z)$.

It is possible for $f_p(z)$ to have a complex zero which we may denote by β_s. Since all the coefficients are real, $\bar{\beta}_s$ will also be a zero of $f_p(z)$, and we denote this zero by β_{s-1}. Under these conditions $s>1$. Let us suppose further that $p>2$. If in $k_3(z)$ we set $\beta=-\beta_s$ and define $f_{p-2}(z)$ by $f_{p-2}(z)=k_3(z)f_p(z)$, then $f_{p-2}(z)$ is regular in $|z|\leqq 1$. Since

$$(3.13) \qquad \Im f_{p-2}(e^{i\theta}) = \frac{(\cos\theta - \cos\nu)(\cos\theta - \cos\mu)}{D_3(\theta, -\beta_s)} \Im f_p(e^{i\theta}),$$

it is possible to select μ and ν so that $f_{p-2}(z) \in T(p-2)$. By the induction hypothesis

$$(3.14) \quad f_{p-2}(z) \ll \frac{z^q}{(1-z)^{2q+2s-4}}\left(\frac{1+z}{1-z}\right)^{2m} \prod_{j=1}^{s-2}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|)$$

and since

$$(3.15) \qquad \frac{1}{k_3(z)} \ll \frac{1}{(1-z)^4} \prod_{j=s-1}^{s}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|)$$

this gives again the bound (1.5).

The preceding work omits three special cases.

(a) There is a single zero at the origin, $q=p=1$, $s=t=0$. Set[4]

$$k_4(z) = (1-z^2)/z, \qquad k_4(e^{i\theta}) = -2i\sin\theta.$$

(b) There is a single real zero $\beta_1 \neq 0$, $s=p=1$, $q=t=0$. Set

[4] In this case the theorem is well known and our proof is essentially that given by Rogosinski [6]. It seems worthwhile, however, to settle all three cases by a parallel argument.

$$k_5(z) = \frac{1 - z^2}{(1 - z/\beta_1)(1 - z\beta_1)}, \qquad k_5(e^{i\theta}) = \frac{-2i \sin \theta}{2 \cos \theta - \beta_1 - \beta_1^{-1}}.$$

(c) There are two complex conjugate roots, $\beta_1 = \bar{\beta}_2$, $s = p = 2$, $q = t = 0$. Set

$$k_6(z) = \frac{(1 - z^2)(1 - 2z \cos \nu + z^2)}{(1 - z/\beta_1)(1 - z\beta_1)(1 - z/\beta_2)(1 - z\beta_2)},$$

$$k_6(e^{i\theta}) = - i \sin \theta(\cos \theta - \cos \nu)/D_3(\theta, -\beta_1).$$

Now in each case $\Im f(z)$ changes sign at $\theta = 0$ and $\theta = \pi$. So if $g(z)$ is defined by

$$(3.16) \qquad g(z) = k_j(z)f(z), \qquad j = 4, 5, 6,$$

it is clear that $g(0) = 1$, and if ν is selected properly in $k_6(z)$, $\Re g(e^{i\theta}) \geqq 0$. Hence in each case $g(z) \ll (1+z)/(1-z)$, and the bound (1.5) follows easily from (3.16) and the form of $k_j(z)$. This completes the proof of (1.5).

Corollary. *Let $f(z)$ satisfy the conditions of Theorem* 1, *and let $F(z)$ be given by* (1.6), *then for* $0 \leqq r < 1$

$$| f^{(j)}(re^{i\theta}) | \leqq F^{(j)}(r), \qquad j = 0, 1, 2, \cdots.$$

This is a trivial consequence of Theorem 1, since all the coefficients in $F(z)$ are positive.

4. **Proof that the bound is sharp.** We shall prove that if t is zero or an odd integer, then the extremal function $F(z)$ given by (1.6) belongs to the set $T(p)$.

It was proved in [2] that if $t = 0$, then $F(z)$ is p-valently starlike, and in this case it obviously belongs to the set $T(p)$.

Now let t be odd. Then $m = [(t+1)/2] = (t+1)/2$. Further, for simplicity let $0 < \beta_j < 1$, $j = 1, 2, \cdots, s$, and let $2\eta_j = \beta_j + \beta_j^{-1}$. The extremal function has the form

$$(4.1) \qquad F(z) = \frac{z^q}{(1 - z)^{2q+2s}} \left(\frac{1 + z}{1 - z}\right)^{t+1} \prod_{j=1}^{s} (1 + 2\eta_j z + z^2).$$

We first observe that $F(z)$ is a rational function of degree $2q + 2s + t + 1$ and hence maps the complex z-plane onto a surface consisting of this number of sheets. A little computation shows that

$$(4.2) \qquad F(e^{i\theta}) = \frac{(-1)^{q+s+m} \sin^{t+1} \theta}{2^q(1 - \cos \theta)^{q+s+t+1}} \prod_{j=1}^{s} (\eta_j + \cos \theta).$$

Let $\epsilon=(-1)^{q+s+m}$ and consider $\Phi(e^{i\theta})=\epsilon F(e^{i\theta})$. If the identity $\sin\theta/(1-\cos\theta)=(1+\cos\theta)/\sin\theta$ is used in (4.2), it is clear that $\Phi(e^{i\theta})$ is a decreasing function for $0<\theta\leqq\pi/2$. Further it is obvious that for $\pi/2\leqq\theta\leqq\pi$, $\Phi(e^{i\theta})$ is also decreasing. Finally, since t is odd, $\Phi(e^{i\theta})$ is an even function. Hence the boundary of the unit circle is mapped by $F(z)$ into a doubly covered slit along the real axis running from the point at infinity into the origin, the direction of the slit being governed by the sign of ϵ. These facts, together with the Schwarz reflection principle, show that $F(z)$ maps $|z|<1$ onto a region consisting of $q+s+m$ sheets, one sheet being slit along the real axis as described, while the remaining sheets are fully covered. Thus $F(z)$ has valence $q+s+m$ and this is less than p for $t>1$.

Let us assume for convenience that $\epsilon=1$ so that the slit is on the positive real axis. The case $\epsilon=-1$ is similar and will be omitted. We shall prove that $F(z)\in T(p)$ by decomposing each circle $|z|=r>\rho$ into four circular arcs and studying the behavior of $\Im F(re^{i\theta})$ on each arc.

Set $S(z)=zF'(z)/F(z)$; then

$$(4.3)\quad S(z)=(q+s)\frac{1+z}{1-z}+2(t+1)\frac{z}{1-z^2}-\sum_{j=1}^{s}\frac{1-z^2}{1+2z\eta_j+z^2},$$

$$(4.4)\quad S(e^{i\theta})=i\left\{(q+s)\frac{\sin\theta}{1-\cos\theta}+\frac{t+1}{\sin\theta}+\sum_{j=1}^{s}\frac{\sin\theta}{\eta_j+\cos\theta}\right\}.$$

It is obvious from (4.1) that $F(z)$ is regular on $|z|=1$, $z\neq\pm1$, and from (4.4) it is clear that on these arcs $F'(z)\neq0$, and the mapping is conformal. Let $\delta>0$ be given, and let C_1 be the arc of the circle $|z|=re^{i\theta}$, $\delta\leqq\theta\leqq\pi-\delta$, and let C_3 be the arc of the same circle $\pi+\delta\leqq\theta\leqq2\pi-\delta$. Let Γ_1 and Γ_3 be the images of C_1 and C_3 respectively under $w=F(z)$. Then the conformality of $F(z)$ shows that for each $\delta>0$ there is a $\rho_1<1$ such that for each r in $\rho_1<r<1$, $\Im(w)<0$ on Γ_1 and $\Im(w)>0$ on Γ_3. Thus, on C_1 and C_3, $\Im F(z)$ does not change sign.

Next let K_2 and K_4 be the circles $|z+1|<\epsilon_2$ and $|z-1|<\epsilon_4$ respectively. For $j=2, 4$, let L_j be the region common to K_j and $|z|<1$, let C_j be the arc of the circle $|z|=r<1$ lying in L_j, and let Γ_j be the image of C_j under $F(z)$. We shall prove that if the circle K_j is sufficiently small, then Γ_j is starlike with respect to the origin. From (4.3), $S(z)$ has an isolated pole of first order at $z=\pm1$, and hence if ϵ_2 and ϵ_4 are sufficiently small, $S^{-1}(z)$ is regular and univalent in K_2 and K_4 and maps each of those regions onto convex regions. But (4.4) shows that $\Re S^{-1}(z)=0$ on $|z|=1$. Hence $\Re S^{-1}(z)$ has constant sign in L_j. Examination of (4.3) for real z shows that in L_2, $\Re S(z)<0$, and that in L_4, $\Re S(z)>0$. The negative sign for $\Re S(z)$ in L_2 means that as z describes C_2 in a counterclockwise direction, $F(z)$ describes Γ_2 in a clockwise direction.

Returning now to $F(z)$, we observe that it has a root of order $t+1$ at $z=-1$ and hence each circle $|z+1|=r<\epsilon_2$ maps into a curve which goes around the origin $t+1$ times. The identity $F(z)=F(z^{-1})$ and equation (4.2) show that the arc of this circle which lies in L_2 maps into a curve which goes around the origin $(t+1)/2$ times. Hence if ρ_2 is sufficiently close to 1, then for each C_2 with $r>\rho_2$, Γ_2 is starlike with respect to the origin, and as z describes C_2 in a counterclockwise direction, Γ_2 goes around the origin slightly less than $(t+1)/2$ times in a clockwise direction, starting from a point in the lower half-plane and ending at the conjugate point in the upper half-plane. Therefore Γ_2 cuts the real axis t times and $\Im F(z)$ changes sign t times on C_2.

At $z=+1$, $F(z)$ has a pole of order $2q+2s+t+1$. An argument similar to the one just given shows that if ρ_4 is sufficiently close to 1, then for $r>\rho_4$, Γ_4 goes around the origin, this time in a counterclockwise direction, slightly less than $q+s+(t+1)/2$ times, and so on C_4, $\Im F(z)$ changes sign $2q+2s+t$ times.

The determination of δ and ρ_1 can now be completed. In the upper half-plane let P_j $(j=2, 4)$ be the point of intersection of $|z|=\rho_j$ with the boundary of K_j and let Q_j be the intersection of the line segment through P_j lying in K_j with the circle $|z|=1$. Then $\arg P_4=\delta_4$, $\arg P_2=\pi-\delta_2$, and $\pi>\delta_j>0$ define δ_j. Then ρ_1 is determined as described earlier, using $\delta=\min\{\delta_2, \delta_4\}$.

Finally let $\rho=\max\{\rho_1, \rho_2, \rho_4\}$. Then for $z=re^{i\theta}$, $\rho<r<1$, $\Im F(z)$ changes sign $2q+2s+2t=2p$ times, and hence $F(z)\in T(p)$.

5. **A conjecture for multivalent functions.** It has been known for some time that if a function is p-valent in $|z|<1$, the magnitude of its power series coefficients depends on the location of its zeros [1], but as far as the author is aware no suggestion has yet been made as to just what the sharp bound is. The work of the preceding sections suggests the following conjecture.

Let $f(z)$ given by (1.3) be regular and p-valent in $|z|<1$, and let $f(z)$ have zeros β_j, $0<|\beta_j|<1$, $j=1, 2, \cdots, s$. Finally let the nonnegative integer t be defined by (1.4) and let

$$(5.1)\quad F_p(z) = \frac{z^q}{(1-z)^{2q+2s}}\left(\frac{1+z}{1-z}\right)^{2t}\prod_{j=1}^{s}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|)$$

$$= z^q + \sum_{n=q+1}^{\infty} B_n z^n,$$

then

$$|b_n| \leqq B_n, \qquad n = q+1, q+2, \cdots . \tag{5.2}$$

It is easy to see that $F_p(z)$ is p-valent for it is a rational function of degree $2q+2s+2t=2p$ and satisfies the equation $F_p(z)=F_p(z^{-1})$.

It may be worth noting that if $u=z/(1-z)^2$, $F_p(z)$ is a polynomial in u. Indeed it is easy to see that

$$F_p(z) = u^q(1+4u)^t \prod_{j=1}^{s} (1+(2+|\beta_j|+|\beta_j|^{-1})u). \tag{5.3}$$

The inequality (5.2) has been proved [2] in the special case that $t=0$, and $F(z)$ is starlike with respect to the origin.

References

1. M. Biernacki, *Sur les fonctions multivalentes d'ordre p*, C. R. Acad. Sci. Paris vol. 203 (1936) pp. 449–451.

2. A. W. Goodman, *On the Schwarz-Christoffel transformation and p-valent functions*, Trans. Amer. Math. Soc. vol. 65 (1950) pp. 204–223.

3. A. W. Goodman and M. S. Robertson, *A class of multivalent functions*, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 127–136.

4. M. S. Robertson, *A representation of all analytic functions in terms of functions with positive real part*, Ann. of Math. vol. 38 (1937) pp. 770–783.

5. ———, *The variation of the sign V for an analytic function U+iV*, Duke Math. J. vol. 5 (1939) pp. 512–519.

6. W. Rogosinski, *Über positive harmonische Entwicklungen und typisch-reele Potenzreihen*, Math. Zeit. vol. 35 (1932) pp. 93–121.

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