COLLINEATIONS IN GENERALIZED SPACES

JACK LEVINE

1. **Introduction.** In this paper we derive simplified forms of the equations defining the infinitesimal projective and affine collineations of a generalized space of paths H_n. The complete solutions of these equations are then obtained for the two-dimensional spaces H_2. It is found that there are nine types of H_2 admitting continuous real groups G_r of projective collineations (P. C.), and also nine types admitting affine collineations (A. C.). In each case the maximum number of parameters r is three. In these classifications all solutions giving H_2 which reduce to ordinary affinely connected or projectively connected spaces are excluded. (These were obtained in [**8**].)[1]

A generalized space of paths H_n of n dimensions is characterized by a set of n functions $H^i(x, dx)$ which are homogeneous of the second degree in $dx^1, \cdots, dx^n$. The paths are defined by

$$d^2x^i/ds^2 + H^i(x, dx/ds) = 0. \tag{1.1}$$

Such spaces have been discussed by Knebelman [**6**] and Douglas [**4**; **5**] and in [**6**] the theory of collineations in H_n has been introduced. Further results on collineations in generalized spaces may be found in [**2**; **7**; **10**; **11**; **12**; **13**; **14**; **15**].

The components of affine connection $\Gamma^i_{jk}(x, dx)$ of an H_n are defined by

$$\Gamma^i_{jk} = \frac{1}{2}\,\frac{\partial^2 H^i}{\partial(dx^j)\partial(dx^k)}, \tag{1.2}$$

and are homogeneous of degree zero in the dx. In terms of the Γ's the paths (1.1) have the familiar form

$$d^2x^i/ds^2 + \Gamma^i_{jk}(dx^j/ds)(dx^k/ds) = 0. \tag{1.3}$$

In case the H^i are second degree polynomials in the dx, the H_n reduces to an ordinary affine space of paths A_n.

Two affine connections $\Gamma^i_{jk}(x, dx)$, $\Gamma'^i_{jk}(x, dx)$ will possess the same paths if

$$\Gamma'^i_{jk} = \Gamma^i_{jk} + \delta^i_j\phi_{\cdot k} + \delta^i_k\phi_{\cdot j} + \phi_{\cdot jk}dx^i, \tag{1.4}$$

Presented to the Society, April 22, 1950 and April 29, 1950; received by the editors April 15, 1950.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

where $\phi(x, dx)$ is an arbitrary function homogeneous of the first degree in the dx [6]. (Subscripts following a dot will indicate differentiations with respect to the dx.)

Elimination of the $\phi_{\cdot i}$ in (1.4) leads to the projective relation $\Pi'^{i}_{jk}=\Pi^{i}_{jk}$, where the components Π^{i}_{jk} of projective connection are defined by

$$(1.5)\quad \Pi^{i}_{jk} \equiv \Gamma^{i}_{jk} - \frac{1}{n+1}(\delta^{i}_{j}\Gamma^{a}_{ak} + \delta^{i}_{k}\Gamma^{a}_{aj} + \Gamma^{a}_{ajk}dx^{i}); \qquad \Pi^{i}_{ij} = 0.$$

The components $\Gamma^{h}_{ijk}\equiv\Gamma^{h}_{ij\cdot k}$ are those of a tensor symmetric in i, j, k.

A collineation in an H_n is defined as a point transformation $\bar{x}^i = \bar{x}^i(x)$ which transforms paths into paths. In case the path-parameter (s) is preserved we have an A. C., otherwise a P. C. A form of the path equations (1.3) unchanged by arbitrary parameter transformations is given by [6],

$$(1.6)\qquad (d^2x^i + \Gamma^{i}_{jk}dx^jdx^k)dx^h - (d^2x^h + \Gamma^{h}_{jk}dx^jdx^k)dx^i = 0.$$

The conditions on an H_n for it to admit a continuous G_r of P. C. were obtained by Knebelman [6], through the use of the infinitesimal P. C. determined by

$$(1.7)\qquad \bar{x}^i = x^i + \xi^i(x)\delta t.$$

These conditions are

$$(1.8)\ \xi^{i}_{;jk} + \xi^{h}B^{i}_{jkh} + \xi^{h}_{;m}\Pi^{i}_{jkh}dx^m - (\delta^{i}_{j}\psi_{;k} + \delta^{i}_{k}\psi_{;j}) = 0,\ (\Pi^{h}_{ijk} \equiv \Pi^{h}_{ij\cdot k}),$$

where a semicolon indicates the (projective) covariant derivative, for example,

$$(1.9)\quad T^{i}_{j;k}(x, dx) = \partial T^{i}_{j}/\partial x^k + T^{h}_{j}\Pi^{i}_{hk} - T^{i}_{h}\Pi^{h}_{jk} - T^{i}_{j\cdot h}\Pi^{h}_{mk}dx^m.$$

Also,

$$(1.10)\qquad \begin{aligned} B^{i}_{jkh} \equiv {} & \frac{\partial\Pi^{i}_{jk}}{\partial x^h} - \frac{\partial\Pi^{i}_{jh}}{\partial x^k} + \Pi^{m}_{jk}\Pi^{i}_{mh} - \Pi^{m}_{jh}\Pi^{i}_{mk} \\ & + (\Pi^{m}_{ak}\Pi^{i}_{jhm} - \Pi^{m}_{ah}\Pi^{i}_{jkm})dx^a, \end{aligned}$$

$$(1.11)\qquad \psi_{;i} = (1/(n+1))\xi^{a}_{;ai}.$$

If an H_n admits the r linearly independent infinitesimal P. C. with generators $X_\alpha f=\xi^{i}_{\alpha|}\,\partial f/\partial x^i$ $(\alpha=1, \cdots, r)$, then it is known to admit the finite continuous (local) group $G_r=[X_1, X_2, \cdots, X_r]$ generated by the $X_\alpha f$ [**2**; **6**; **12**].

It has been shown [**2**; **12**] that (1.8) can be expressed in the form

$$\Delta \Pi^{i}_{jk} = 0, \tag{1.12}$$

where Δ is the Lie derivative operator

$$\Delta \Pi^{i}_{jk} = \lim_{\delta t \to 0} \frac{\Pi^{i}_{jk}(\bar{x}, d\bar{x}) - \overline{\Pi}^{i}_{jk}(\bar{x}, d\bar{x})}{\delta t} \tag{1.13}$$

with respect to the infinitesimal transformation (1.7). For a general discussion of this operator, references [**3**; **11**; **12**; **14**; **15**] may be consulted.

When written in expanded form (1.8) or (1.12) is

$$\begin{aligned} &\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} + \Pi^{i}_{jm} \frac{\partial \xi^m}{\partial x^k} + \Pi^{i}_{mk} \frac{\partial \xi^m}{\partial x^j} - \Pi^{m}_{jk} \frac{\partial \xi^i}{\partial x^m} + \Pi^{i}_{jkm} \frac{\partial \xi^m}{\partial x^h} dx^h \\ &\qquad + \xi^m \frac{\partial \Pi^{i}_{jk}}{\partial x^m} = \frac{1}{n+1}\left(\delta^{i}_{j} \frac{\partial^2 \xi^h}{\partial x^h \partial x^k} + \delta^{i}_{k} \frac{\partial^2 \xi^h}{\partial x^h \partial x^j} \right). \end{aligned} \tag{1.14}$$

If in (1.14) the Π^i_{jk} be replaced by Γ^i_{jk} and the right member by 0, we obtain the equations of A. C.

$$\begin{aligned} &\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} + \Gamma^{i}_{jm} \frac{\partial \xi^m}{\partial x^k} + \Gamma^{i}_{mk} \frac{\partial \xi^m}{\partial x^j} - \Gamma^{m}_{jk} \frac{\partial \xi^i}{\partial x^m} \\ &\qquad + \xi^m \frac{\partial \Gamma^{i}_{jk}}{\partial x^m} + \Gamma^{i}_{jkm} \frac{\partial \xi^m}{\partial x^h} dx^h = 0. \end{aligned} \tag{1.15}$$

These latter equations may also be written in terms of the Lie derivative as [**3**]

$$\Delta \Gamma^{i}_{jk} = 0. \tag{1.16}$$

2. **Simplified form of collineation equations.** To obtain a simplified form of these equations we consider first (1.12) and define quantities $P^i(x, dx)$ by

$$P^i \equiv H^i - \frac{1}{n+1} F dx^i, \tag{2.1}$$

with

$$F(x, dx) = H^{i}_{\cdot i} \tag{2.2}$$

being homogeneous of first degree and P^i of second degree in the dx.

It is then easily shown that

$$2\Pi^{i}_{jk} = P^{i}_{\cdot jk} \tag{2.3}$$

and

$$2\Delta\Pi^{i}_{jk} = \Delta(P^{i}_{\cdot jk}) = (\Delta P^{i})_{\cdot jk}, \tag{2.4}$$

since the operators Δ and $\partial(dx)$ are here commutative [**2**; **3**; **10**; **11**].

It follows from the homogeneity in dx of $\Delta\Pi^{i}_{jk}$ that

$$\Delta P^{i} = (\Delta\Pi^{i}_{jk})dx^{j}dx^{k}, \tag{2.5}$$

and thus the equations of P. C. can be expressed in the equivalent form

$$\Delta P^{i} = 0. \tag{2.6}$$

Equations (2.5) show also that ΔP^{i} are components of a contravariant vector.

In a similar manner it can be shown that the A. C. equations can be expressed as

$$\Delta H^{i} = 0, \tag{2.7}$$

with ΔH^{i} also contravariant vector components.

To obtain the explicit form of ΔP^{i} we may use (1.4) and (2.5) or else the definition $\Delta P^{i} = \lim_{\delta t\to 0}\,[P^{i}(\bar{x},\ d\bar{x}) - \overline{P}^{i}(\bar{x},\ d\bar{x})]/\delta t$. In the latter case we make use of the transformation equation [**4**, **12**]

$$\overline{H}^{i} = H^{j}\frac{\partial\bar{x}^{i}}{\partial x^{j}} - \frac{\partial^{2}\bar{x}^{i}}{\partial x^{j}\partial x^{k}}dx^{j}dx^{k}, \tag{2.8}$$

$$\overline{F} = F - 2\frac{\partial^{2}\bar{x}^{i}}{\partial x^{j}\partial x^{k}}\frac{\partial x^{k}}{\partial\bar{x}^{i}}dx^{j}. \tag{2.9}$$

There is obtained

$$\begin{aligned}\Delta P^{i} = {} & \frac{\partial^{2}\xi^{i}}{\partial x^{j}\partial x^{k}}u^{j}u^{k} + \xi^{m}\frac{\partial P^{i}}{\partial x^{m}} + \frac{\partial\xi^{m}}{\partial x^{h}}\frac{\partial P^{i}}{\partial u^{m}}u^{h} - \frac{\partial\xi^{i}}{\partial x^{m}}P^{m} \\ & - \frac{2}{n+1}\frac{\partial^{2}\xi^{h}}{\partial x^{h}\partial x^{a}}u^{a}u^{i},\end{aligned} \tag{2.10}$$

where $u^{i} = dx^{i}$.

It is to be noted that the P^{i} satisfy the identity

$$P^{i}_{\cdot i} = 0. \tag{2.11}$$

Also, corresponding to the projective change (see (1.4))

$$H'^i = H^i + \phi dx^i, \qquad F' = F + (n+1)\phi, \tag{2.12}$$

we have

$$P'^i = P^i. \tag{2.13}$$

To obtain the explicit form of ΔH^i we proceed as above, making use of (2.8). We find

$$\Delta H^i = \frac{\partial^2 \xi^i}{\partial x^j \partial x^k} u^j u^k + \frac{\partial H^i}{\partial x^m} \xi^m + \frac{\partial H^i}{\partial u^m} \frac{\partial \xi^m}{\partial x^h} u^h - \frac{\partial \xi^i}{\partial x^m} H^m. \tag{2.14}$$

This expression has also been obtained by Kosambi [7] in a different way.

It follows from (2.1) that in order for a P. C. to reduce to an A. C. the ξ^i must satisfy besides (2.6) the further condition

$$\Delta F = \xi^i \frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial u^i} \frac{\partial \xi^i}{\partial x^j} u^j + 2 \frac{\partial^2 \xi^i}{\partial x^i \partial x^j} u^j = 0. \tag{2.15}$$

The A. C. can thus be obtained either by solving (2.7) directly or else by solving the system (2.6) and (2.15). Since we wish to find the P. C. also we have used the latter method.

3. **Collineations in** H_2. The results of the preceding section are now applied to the two-dimensional spaces H_2. Equations (2.6), considered in the unknowns P^i, are solved to obtain all spaces H_2 which admit real groups G_r of P. C. As in [8] the ξ^i are obtained from the classification of Lie [9] giving all real continuous groups G_r in two variables. We also put $x^1, x^2 = x, y$ and $u^1, u^2 = u, v$.

Now if $n=2$, (2.11) shows that $dP = -P^2 du + P^1 dv$ is an exact differential with

$$P^1 = \partial P/\partial v, \qquad P^2 = -\partial P/\partial u. \tag{3.1}$$

Also, P is homogeneous of the third degree in u, v. The Π^i_{jk} can then be expressed in terms of P which remains as the basic function to be determined in each case. If it is desired to obtain the H^i, we use (2.1) in which F is now considered as an arbitrary function homogeneous of the first degree in the dx^i.

To find the A. C. determined by a particular G_r we solve (2.15) for F, and using the solutions for P^1, P^2 as given by (3.1) through the P. C. solution we obtain the H^i from (2.1).

To illustrate the general procedure we find the spaces admitting the G_3 $[p, q, (y+cx)p+(cy-x)q]$ (c an arbitrary constant) as a group of collineations. Here, as later, $p=\partial f/\partial x$, $q=\partial f/\partial y$.

We obtain first the P. C. The generators $X_1 f=p$, $X_2 f=q$ show by (2.6) that P^i is a function of u and v only. On substituting the third generator components $\xi^1_{3|}=y+cx$, $\xi^2_{3|}=cy-x$ in (2.6) we have

$$v\frac{\partial P^1}{\partial u}-u\frac{\partial P^1}{\partial v}=-cP^1+P^2, \tag{3.2a}$$

$$v\frac{\partial P^2}{\partial u}-u\frac{\partial P^2}{\partial v}=-P^1-cP^2, \tag{3.2b}$$

with the solutions

$$\begin{aligned} P^1 &= u^2(fw+g)(1+w^2)^{1/2}e^{c\tan^{-1}w}, \\ P^2 &= u^2(gw-f)(1+w^2)^{1/2}e^{c\tan^{-1}w}, \qquad w=v/u, \end{aligned} \tag{3.3}$$

where f, g are arbitrary constants not both zero.

From (3.1) we then obtain $3g-cf=0$, and the value of P,

$$P=\frac{f}{3}u^3(1+w^2)^{1/2}e^{c\tan^{-1}w}=\frac{f}{3}(dx^2+dy^2)^{1/2}e^{c\tan^{-1}dy/dx}, \qquad f\neq 0. \tag{3.4}$$

The components Π^i_{jk} of projective connection of the corresponding H_2 can now be found by (2.3) and (3.3) with $g=cf/3$.

To determine the spaces admitting the above G_3 as A. C. we must solve (2.15) for F. From X_1 and X_2 we have $F(u, v)$, and X_3 gives the single condition

$$v\frac{\partial F}{\partial u}-u\frac{\partial F}{\partial v}+cF=0. \tag{3.5}$$

Hence $F=du(1+w^2)^{1/2}\,e^{c\tan^{-1}w}$ with d an arbitrary constant.

The values of H^1 and H^2 of the H_2 admitting the G_3 as A. C. are thus

$$\begin{aligned} H^1 &= (a\,dy+b\,dx)(dx^2+dy^2)^{1/2}e^{c\tan^{-1}dy/dx}, \\ H^2 &= (b\,dy-a\,dx)(dx^2+dy^2)^{1/2}e^{c\tan^{-1}dy/dx}. \end{aligned} \tag{3.6}$$

The affine connection components can then be found from (1.2).

In the following two lists which represent all possible H_2 admitting real groups of P. C. and A. C. respectively the reducible H_2 have been excluded. In the case of P. C. such spaces result from functions P which are cubic polynomials in dx, dy, so that the Π^i_{jk} are functions of position (x and y) only.

The reducible H_2 admitting A. C. are such that the H^i are second-degree polynomials in dx, dy.

Each of the 9 groups in the two lists is a complete group of collineations, that is, it is the group of maximum number of parameters which the associated H_2 will admit. For each of the six G_3 spaces this follows, since otherwise the H_2 would be reducible. That the G_1 and G_2 groups are complete follows from similar results of [**8**].

H_2 spaces admitting complete groups of projective collineations

[G1.1] $[p]$,
$$P = P(y; dx, dy).$$

[G2.1] $[p, q]$,
$$P = P(dx, dy).$$

[G2.2] $[p, xp + yq]$,
$$P = (1/y)Q(dx, dy).$$

[G3.1] $[p, q, xp + cyq]$ $(c \neq 0, 1/2, 1, 2)$,
$$P = k[(c-1)/(1-2c)](dx)^3(dy/dx)^{(2-c)/(1-c)}.$$

[G3.2] $[p, q, xp + (x+y)q]$,
$$P = k(dx)^3 e^{-dy/dx}.$$

[G3.3] $[p, xp + yq, x^2p + (2xy + y^2)q]$,
$$P = (dx)^3\left(1 + \frac{dy}{dx}\right)\left[k\left(1 + \frac{dy}{dx}\right)^{1/2} + \frac{2}{3}\left(2 + \frac{dy}{dx}\right)\right].$$

[G3.4] $[p, q, (y + cx)p + (cy - x)q]$,
$$P = k(dx^2 + dy^2)^{1/2} e^{c \tan^{-1} dy/dx}.$$

$\begin{bmatrix}\text{G3.5}\\ \text{G3.6}\end{bmatrix}$ $[yp - xq, (1 + \epsilon x^2)p + \epsilon xyq, \epsilon xyp + (1 + \epsilon y^2)q]$ $(\epsilon = \pm 1)$,
$$P = k\left[\frac{\epsilon(dx^2 + dy^2) + (x\,dy - y\,dx)^2}{x^2 + y^2 + \epsilon}\right]^{3/2}.$$

In [G1.1] and [G2.1] P is an arbitrary function of its arguments as is Q in [G2.2] (homogeneous of third degree in dx, dy), and in all other cases k is an arbitrary nonzero constant.

H_2 spaces admitting complete groups of affine collineations

[G1.1] $H^i = H^i(y; dx, dy)$.

[G2.1] $H^i = H^i(dx, dy)$.

[G2.2] $H^i = (2/y)A^i(dx, dy)$.

[G3.1] $$H^1 = a(dx)^2(dy/dx)^{1/(1-c)},\ H^2 = b\,dx\,dy(dy/dx)^{1/(1-c)} \quad (c \neq 0, 1).$$

[G3.2] $$H^1 = 2a(dx)^2e^{-dy/dx}, \qquad H^2 = 2(dx)^2[a(dy/dx) + b]e^{-dy/dx}.$$

[G3.3] $$H^1 = (2/y)(dx)^2(1 + 2aR),$$ $$H^2 = -(2/y)(dx)^2(1+2aR+2bR^3+2R^4),\ R = \left(1 + \frac{dy}{dx}\right)^{1/2}.$$

[G3.4] $$H^1 = (a\,dy + b\,dx)(dx^2 + dy^2)^{1/2}e^{c \tan^{-1} dy/dx},$$ $$H^2 = (b\,dy - a\,dx)(dx^2 + dy^2)^{1/2}e^{c \tan^{-1} dy/dx}.$$

$\begin{bmatrix} \text{G3.5} \\ \text{G3.6} \end{bmatrix}$
$$H^1 = (dx)^2\left[-2\,\frac{x + y(dy/dx)}{x^2 + y^2 + \epsilon} + \frac{1}{(x^2+\epsilon)^{1/2}}\,\frac{1}{(x^2+y^2+\epsilon)^{1/2}}\,(aT + b)(1 + T^2)^{1/2}\right],$$
$$H^2 = (dx)^2\left[-2\,\frac{x + y(dy/dx)}{x^2 + y^2 + \epsilon}\,\frac{dy}{dx} + \frac{(aT + b)(1 + T^2)^{1/2}}{(x^2 + \epsilon)^{3/2}}\left(\frac{bT - a}{aT + b} + \frac{xy}{(x^2 + y^2 + \epsilon)^{1/2}}\right)\right],$$
$$T = [(x^2 + \epsilon)(dy/dx) - xy]/(x^2 + y^2 + \epsilon)^{1/2}.$$

Here a, b, c are arbitrary constants, with a, b not both zero. Also, in [G3.1] $b \neq 0$ if $c = 1/2$, and $a \neq 0$ if $c = 2$. The A^i in [G2.2] are arbitrary functions homogeneous of second degree in dx, dy. Similar remarks apply to the H^i of [G1.1] and [G2.1].

We summarize these results in the following theorem:

THEOREM 3.1. *If a (nonreducible) two-dimensional generalized space H_2 admits a real continuous group G_r of projective or affine collineations then $r \leqq 3$. There are nine such complete groups of collineations, one G_1, two G_2, and six G_3.*

BIBLIOGRAPHY

1. Ludwig Berwald, *Untersuchung der Krummung allgemeiner metrischer Raume auf Grund des in ihnen herrschenden Parallelismus*, Math. Zeit. vol. 25 (1928) pp. 40–73.

2. R. S. Clark, *Projective collineations in a space of K-spreads*, Proc. Cambridge Philos. Soc. vol. 41 (1945) pp. 210–223.

3. E. T. Davies, *On the isomorphic transformations of a space of K-spreads*, J. London Math. Soc. vol. 18 (1943) pp. 100–107.

4. Jesse Douglas, *The general geometry of paths*, Ann. of Math. vol. 29 (1928) pp. 154–166.

5. ———, *System of K-dimensional manifolds in an N-dimensional space*, Math. Ann. vol. 105 (1931) pp. 527–564.

6. M. S. Knebelman, *Collineations and motions in generalized spaces*, Amer. J. Math. vol. 51 (1928) pp. 527–564.

7. D. D. Kosambi, *Collineations in path-space*, J. Indian Math. Soc. N.S. vol. 1 (1934) pp. 69–72.

8. Jack Levine, *Classification of collineations in projectively and affinely connected spaces of two dimensions*, Ann. of Math. vol. 52 (1950) pp. 465–477.

9. Sophus Lie and Friedrich Engel, *Theorie der Transformationsgruppen*, vol. 3, Leipzig, Teubner, 1893, pp. 71–73, 379–380.

10. Buchin Su, *On the isomorphic transformations of K-spreads in a Douglas space*. I, Science Record vol. 2 (1947) pp. 11–19.

11. ———, *On the isomorphic transformations of K-spreads in a Douglas space*. II, Science Record vol. 2 (1948) pp. 139–146.

12. ———, *Descriptive collineations in spaces of K-spreads*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 495–507.

13. ———, *A characteristic property of affine collineations in a space of K-spreads*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 136–138.

14. Kentaro Yano, *Lie derivatives in general space of paths*, Proc. Imp. Acad. Tokyo vol. 21 (1945) pp. 363–371.

15. ———, *Groups of transformations in generalized spaces*, Tokyo, Akademeia Press, 1949.

North Carolina State College

ON THE CONTINUITY OF PARAMETRIC LINEAR OPERATIONS[1]

B. J. PETTIS

The proofs of the theorems asserting strong continuity for semigroups of linear operations in Banach spaces usually involve measurability and integrability of Banach-space-valued functions [2, pp. 183–184].[2] A theorem of this type in which the assumptions and the proof are purely topological is given below.

Let G be both a topological space and an additive group, and let H be a subset of G. For each h in H let $D(h)$ be all points g in H satisfying these two conditions: (1) $h-g\in H$, (2) for each open set N_g about g there is an open set N_h about h such that $h-g+(H\cap N_g) \supset H\cap N_h$. Letting X be a complex linear normed space, a set $\Gamma=[\gamma]$ of bounded complex linear functionals on X is *a total set for E*, E a subset of X, if $\|x\|=\sup\ [|\gamma(x)|,\ \gamma\in\Gamma]$ holds for every x in the smallest linear subspace containing E. A function T_h on H to the

Presented to the Society, April 21, 1951; received by the editors May 29, 1950.

[1] This paper was written under Contract N7-onr-434, Task Order III, Navy Department (The Office of Naval Research).

[2] Numbers in brackets refer to the references at the end of the paper.