NOTE ON THE HURWITZ ZETA-FUNCTION[1]

N. J. FINE

It is well known that Riemann[2] gave two proofs of the functional equation for $\zeta(s)$, the first depending on a contour integration, the second on the transformation equation for $\vartheta_3(0|\tau)$. Hurwitz[3] introduced his generalized zeta-function, defined for $R(s)>1$ and $0<a<1$ by[4]

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}, \tag{1}$$

showed that it can be continued to the entire s-plane with the exception of a simple pole at $s=1$, and proved that for $R(s)>1$,

$$\begin{aligned} \zeta(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s}\Bigg\{\cos\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi a}{n^s} \\ + \sin\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\sin 2n\pi a}{n^s}\Bigg\}. \end{aligned} \tag{2}$$

His method of proof depends on a contour integral and parallels Riemann's first proof. It appears to have been overlooked that the second method of Riemann can be generalized to obtain the same results.[5] The purpose of this paper is to supply such a proof.

For $0<a<1$ and $x>0$, define

$$\begin{aligned} f(a, x) &= \vartheta_3(\pi a, ix) \\ &= 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} \cos 2\pi n a. \end{aligned} \tag{3}$$

By the transformation equation for the ϑ-function,[6]

Received by the editors June 10, 1950.

[1] This work is an offshoot of investigations carried out under the auspices of the Office of Naval Research, Contract N9-ONR90,000.

[2] B. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Preussischen Akademie der Wissenschaften (1859, 1860) pp. 671–680.

[3] A. Hurwitz, Zeitschrift für Mathematik und Physik vol. 27 (1882) p. 95.

[4] Throughout this paper, $x^s = \exp(s \log x)$, the logarithm being real for $x>0$.

[5] R. Lipschitz (J. Reine Angew. Math. vol. 105, pp. 127–159) has used the theta-function transformation device to derive a functional equation for a general type of zeta-function, but his results do not appear to include ours.

[6] Whittaker and Watson, *Modern analysis*, p. 475.

$$f(a, x^{-1}) = x^{1/2}e^{-\pi a^2 x}f(iax, x) = x^{1/2}\sum_{n=-\infty}^{+\infty} e^{-\pi x(n+a)^2}. \tag{4}$$

From (3) and (4) it is easy to see that $f(a, x)$ tends to the values 1 and 0 exponentially as x tends to ∞ and 0, respectively. Hence the two functions

$$F(a, s) = \int_0^1 f(a, x)x^{s/2-1}dx, \tag{5}$$

$$G(a, s) = \int_1^\infty (f(a, x) - 1)x^{s/2-1}dx \tag{6}$$

are entire in s. We define

$$H(a, s) = F(a, s) + G(a, s) - 2/s. \tag{7}$$

For a later purpose, we observe that

$$\frac{\partial}{\partial a}H(a, s) = \int_0^\infty \frac{\partial}{\partial a}f(a, x)x^{s/2-1}dx \tag{8}$$

is also an entire function of s.

Now for $R(s) > 1$,

$$H(a, s) = \int_0^\infty (f(a, x) - 1)x^{s/2-1}dx$$

$$= 2\sum_{n=1}^{\infty} \cos 2\pi na \int_0^\infty e^{-\pi n^2 x}x^{s/2-1}dx,$$

$$H(a, s) = 2\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{\cos 2\pi na}{n^s} \qquad (R(s) > 1). \tag{9}$$

For $R(s) < 0$,

$$G(a, s) = \frac{2}{s} + \int_1^\infty f(a, x)x^{s/2-1}dx,$$

so that

$$H(a, s) = \int_0^\infty f(a, x)x^{s/2-1}dx$$

$$= \int_0^\infty f(a, x^{-1})x^{-s/2-1}dx.$$

By (4), therefore,

$$H(a, s) = \sum_{n=-\infty}^{+\infty} \int_0^\infty e^{-\pi x(n+a)^2} x^{(1-s)/2-1} dx,$$

$$(10) \quad H(a, s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right) \sum_{n=-\infty}^{+\infty} | n + a |^{-(1-s)} \qquad (R(s) < 0).$$

With s replaced by $1-s$, (10) becomes

$$(11) \quad \begin{aligned} H(a, 1-s) &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\left\{ \sum_{n=0}^{\infty} (n+a)^{-s} + \sum_{n=0}^{\infty} (n+1-a)^{-s} \right\} \\ &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\{\zeta(s, a) + \zeta(s, 1-a)\} \qquad (R(s) > 1). \end{aligned}$$

Now it is easy to see, from (1), that

$$\frac{\partial}{\partial a}\zeta(s, a) = - s\zeta(s+1, a).$$

Hence, differentiating (11) and replacing s by $s-1$, we have

$$(12) \quad \begin{aligned} \frac{\partial}{\partial a} H(a, 2-s) = &- (s-1)\pi^{-(s-1)/2}\Gamma\left(\frac{s-1}{2}\right)\{\zeta(s, a) \\ &- \zeta(s, 1-a)\} \qquad (R(s) > 2). \end{aligned}$$

Combining (11) and (12) yields

$$(13) \quad \begin{aligned} 2\zeta(s, a) = &\frac{\pi^{s/2}}{\Gamma(s/2)} H(a, 1-s) \\ &- \frac{\pi^{(s-1)/2}}{(s-1)\Gamma((s-1)/2)} \frac{\partial}{\partial a} H(a, 2-s) \quad (R(s) > 2). \end{aligned}$$

Equation (13) provides the analytic continuation of $\zeta(s, a)$, and shows that $\zeta(s, a) - (s-1)^{-1}$ is an entire function of s. Replace s by $1-s$ in (13):

$$(14) \quad \begin{aligned} 2\zeta(1-s, a) = &\frac{\pi^{(1-s)/2}}{\Gamma((1-s)/2)} H(a, s) \\ &+ \frac{\pi^{-s/2}}{s\Gamma(-s/2)} \frac{\partial}{\partial a} H(a, 1+s). \end{aligned}$$

For $R(s) > 1$, we may use (9) to get

$$(15) \quad \frac{\partial}{\partial a} H(a, 1+s) = - 4\pi^{(1-s)/2}\Gamma\left(\frac{1+s}{2}\right) \sum_{n=1}^{\infty} \frac{\sin 2\pi n a}{n^s}.$$

Substituting from (9) and (15) into (14), and simplifying by well known formulas for the Γ-function, we obtain the desired relation (2).

The proof just presented does not cover the classical Riemann zeta-function, defined, for $R(s)>1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

But it is easy to see that, for $R(s)>1$,

$$\begin{aligned} \zeta\left(s, \frac{1}{2}\right) &= \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\right)^{-s} = 2^s \sum_{n \text{ odd}} n^{-s} \\ &= 2^s\left\{\sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{\infty}(2n)^{-s}\right\} \\ &= (2^s - 1)\zeta(s). \end{aligned} \tag{16}$$

This provides the continuation of $\zeta(s)$ and shows that it has a simple pole at $s=1$, with possible simple poles at $s=2n\pi i/\log 2$ ($n=0, \pm 1, \pm 2, \cdots$). Now if we set $a=1/2$ in equation (2), we obtain, for $R(s)>1$,

$$\begin{aligned} \zeta\left(1-s, \frac{1}{2}\right)\frac{(2\pi)^s}{2\Gamma(s)\cos(\pi s/2)} &= \sum_{n=1}^{\infty}(-1)^n n^{-s} = \sum_{n \text{ even}} n^{-s} - \sum_{n \text{ odd}} n^{-s} \\ &= 2\sum_{n=1}^{\infty}(2n)^{-s} - \sum_{n=1}^{\infty} n^{-s} \\ &= (2^{1-s} - 1)\zeta(s). \end{aligned}$$

Using (16) with s replaced by $1-s$, we get the required functional equation for $\zeta(s)$,

$$\zeta(1-s) = \frac{2\Gamma(s)\cos(\pi s/2)}{(2\pi)^s}\zeta(s). \tag{17}$$

Since the right side of (17) is regular at $s=1-2n\pi i/\log 2$, the only singularity of $\zeta(s)$ is the simple pole at $s=1$.

UNIVERSITY OF PENNSYLVANIA