NIL PI-RINGS

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Let S be a ring which satisfies a polynomial identity (in short: a PI-ring). The ring S will be said to be a PI-ring of degree d if d is the minimal degree of the polynomial identities satisfied by S. We denote by $N=N(S)$ the radical of S, that is, the sum of all nilpotent ideals of S. Levitzki [1][1] has proved that a nil PI-ring of degree d is an L-ring (that is, it coincides with its lower radical) and its length is bounded by $\log d/\log 2$. In the present note we show that the length of a nil PI-ring is not greater than 2 and that nil PI-rings of length 2 really exist. Even more, if S is a nil PI-ring of degree d then S/N is a nilpotent ring whose index is bounded by $[d/2]$. This is a direct consequence of the following generalization of [1, Theorem 1]:

THEOREM 1. *If S is a* PI-*ring*[2] *of degree d and T is a nil subring of S, then $T^m \subseteq N$ where $m=[d/2]$.*

PROOF. First we consider the case[3] where T is a nilpotent subring of S. The proof of this case differs from the proof of [1, Theorem 1] only in that we consider a nilpotent subring instead of a single nilpotent element. That is, we consider the following subrings of S:

$$\begin{aligned} A_{2i-1} &= T^{n-i}ST^{i-1}, \\ A_{2i} &= T^{n-i}ST^{i}, \end{aligned} \qquad i = 1, 2, \cdots, n, \tag{1}$$

where n is an integer greater than $m=[d/2]$.

It is readily seen that $A_\lambda A_\mu \subseteq ST^nS$ if $\lambda > \mu$, hence

$$A_{i_1}A_{i_2}\cdots A_{i_d} \subseteq ST^nS, \tag{2}$$

if $(i_1, \cdots, i_d)$ is a permutation of the d letters $1, 2, \cdots, d$ which is not the identical permutation.

By (1) it follows also that

$$A_1A_2\cdots A_d = (T^{n-1}S)^dT^m. \tag{3}$$

We may assume by [1, Lemma 3] that S satisfies the following identity:[4]

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] It is not assumed that S is a nil ring.

[3] I am indebted to Levitzki for the present proof of this case.

[4] Compare with (10) of [1]. For the conditions satisfied by the coefficients of the identity see [1, p. 335].

$$(4) \qquad x_1x_2 \cdots x_d = \sum \beta^{-1}\beta_{(i)}x_{i_1} \cdots x_{i_d},$$

where the sum ranges over all permutations (i) of d letters, except the identical permutation.

From (2), (3), and from condition (II) of [1] satisfied by the coefficients of the identity (1), it follows by substituting $x_i = a_i$ in (4) where a_i ranges over all elements of A_i, $i=1, \cdots, d$, that

$$(5) \qquad (T^{n-1}S)^dT^m \subseteq ST^nS.$$

Since T is nilpotent, there exists a smallest exponent n such that ST^nS is a nilpotent ideal. Suppose $n>m$, then by (5) it follows easily that $(ST^{n-1}S)^{d+1} \subseteq ST^nS$, hence $ST^{n-1}S$ is also nilpotent which is a contradiction to the minimality of n. This completes the proof of the theorem in the case of nilpotent subrings.

We turn now to the general case. Let T be a nil subring of S. By [1, Theorem 1][5] it follows that the quotient ring $(T, N)/N$ satisfies an identity of the form $x^m=0$, hence $(T, N)/N$ is semi-nilpotent (Kaplansky [3, Theorem 5] and Levitzki [1]). Since N is semi-nilpotent, the subring (T, N) of S is also semi-nilpotent. Let $t_1, \cdots, t_m$ be any m elements of T, then the semi-nilpotency of T implies that the ring $\{t_1, \cdots, t_m\}$ generated by these elements is nilpotent, hence by the preceding case $\{t_1, \cdots, t_m\}^m \subseteq N$. Thus $t_1 \cdot t_2 \cdots t_m \in N$. Since this holds for any arbitrarily chosen elements of T, we have $T^m \subseteq N$. q.e.d.

Remark. By the preceding proof it follows that if T is a nilpotent subring of S of index $p>m$, then ST^mS is a nilpotent ideal in S. A more detailed application of (5) shows that $1+d+\cdots+d^{p-m}$ is an upper bound for the index of ST^mS. Indeed by (5) we have $(ST^mS)^{d+1} \subseteq ST^{m+1}S$ and for the same reason $(ST^{m+1}S)^dST^m \subseteq ST^{m+2}S$; hence $(ST^mS)^{1+d+d^2} \subseteq ST^{m+2}S$. By a successive application of (5), we obtain

$$(ST^mS)^{1+d+\cdots+d^{p-m}} \subseteq ST^pS = 0,$$

which proves the remark.

By the preceding theorem, we have the following corollary.

Corollary. *If S is a* PI-*ring of degree d such that its radical N is a nilpotent ideal of index p, then the nil subrings of S are nilpotent rings of index not greater than $p[d/2]$.*

Remark. It has been shown recently [2] that the total matric algebra of order n^2 over a commutative field is a PI-ring of degree

[5] Apparently, [1, Theorem 1] is a special case of the preceding case of our theorem.

$2n$. Hence by the preceding corollary it follows that the nil subrings of such algebras are nilpotent rings of index less than or equal to n. This is a special case of the well known result concerning nil subrings of rings which satisfy both chain conditions.

A simple consequence of Theorem 1 is:

THEOREM 2. *If S is a nil* PI-*ring of degree d, then S/N is a nilpotent ring whose index is bounded by* $[d/2]$.

This implies that in this case S is a nil PI-ring, $S=N_2(S)$, that is:

COROLLARY. *A nil* PI-*ring is an* L-*ring of length less than or equal to* 2.

We conclude with an example of a nil PI-ring S of degree $2n$ such that S/N is a nilpotent ring whose index is n. This example shows that Theorem 2 provides a complete solution of the problem of the length of nil PI-rings and their structure modulo their radical.

We construct our example as follows: Let R be a commutative ring with a unit such that its radical $N(R)$ is not nilpotent. Denote by c_{ik}, $i, k=1, 2, \cdots, n$, an orthogonal base of a total matric algebra R_n of order n^2 over R. One can easily generalise [2, Theorem 1] to total matric algebras over commutative rings and thus one obtains the result that R_n satisfies a polynomial identity of degree $2n$ (that is, the standard identity $S_{2n}(x)=0$). Our required ring S is defined as the totality of the matrices $\sum\alpha_{ik}c_{ik}$ where $\alpha_{ik}\in N(R)$ for $i\geqq k$. No restriction is imposed on the elements α_{ik}, $i<k$, except that $\alpha_{ik}\in R$. Since $S\subseteq R_n$, it follows that S is a PI-ring of degree less than or equal to $2n$. It is readily verified that $S^n\subseteq N(R)_n\subseteq N(S)$ where $N(R)_n$ is the totality of the matrices $\sum\beta_{ik}c_{ik}$, $\beta_{ik}\in N(R)$. Now consider the element $c=c_{12}+c_{23}+\cdots+c_{n-1\,n}$. The ideals c^iS, $i=1, 2, \cdots, n-1$, contain a ring isomorphic to $N(R)$, that is, the ring of the matrices $c^i\bar{r}=\rho c_{11}$ where $\bar{r}=\rho c_{i+1\,1}$, $\rho\in N(R)$. The latter ring is not nilpotent, hence $c^i\notin N(S)$. This implies by [1, Theorem 1] that the degree of S is greater than or equal to $2n$. This completes the proof that the ring S has the required properties.

BIBLIOGRAPHY

1. J. Levitzki, *A theorem on polynomial identities*, Proceedings of the American Mathematical Society vol. 1 (1950) pp. 334–341.

2. S. Amitsur and J. Levitzki, *Remarks on minimal identities for algebras*, Proceedings of the American Mathematical Society vol. 1 (1950) pp. 449–463.

3. I. Kaplansky, *Rings with a polynomial identity*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 575–580.

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