AVERAGES OF THE COEFFICIENTS OF SCHLICHT FUNCTIONS

G. MILTON WING

We shall consider throughout this paper a function

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad a_1 = 1,$$

analytic and schlicht in the unit circle. According to a classical conjecture of Bieberbach, $|a_n| \leqq n$. Recently Bazilevič has proved that $\limsup_{n\to\infty} |a_n|/n \leqq e/2$ [1].[1] Our interest lies in the average behavior of the coefficients. It is clear that if the conjecture holds, then

$$\left| \sum_{j=1}^{n} a_j \right| / C_{n+1,2} \leqq 1.$$

More generally, let us define

$$S_n(k) = \sum_{j=0}^{n-1} C_{j+k-1,k-1} a_{n-j} \qquad (k \geqq 1)$$

and

$$\sigma_n(k) = |S_n(k)| / C_{n+k,k+1}.$$

If $|a_n| \leqq n$, then

$$|S_n(k)| \leqq \sum_{j=0}^{n-1} C_{j+k-1,k-1}(n-j) = C_{n+k,k+1}$$

so that $\sigma_n(k) \leqq 1$. It is easy to see that the result of Bazilevič implies that

$$\limsup_{n\to\infty} \sigma_n(k) \leqq \frac{e}{2}.$$

We prove two theorems concerning the averages $\sigma_n(k)$. Using only classical results we obtain a bound on $\limsup_{n\to\infty} \sigma_n(k)$ and show that this bound tends to unity for large k. By applying recent information concerning the map of the circle $|z| = r < 1$ by the function $f(z)$, we get estimates on $\limsup_{n\to\infty} \sigma_n(k)$ for small k.

Presented to the Society, June 17, 1950; received by the editors August 11, 1950.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

THEOREM 1. *Let $k>1$. Then*

$$\limsup_{n\to\infty} \sigma_n(k) \leqq \frac{e^{k+1}\Gamma(k+2)\Gamma(k-1)}{(k+1)^{k+1}2^{k-1}\Gamma^2(k/2)} = A(k),$$

and $\lim_{k\to\infty} A(k) = 1$.

PROOF. We write

$$(1)\qquad \begin{aligned} S_n(k) &= \frac{1}{2\pi i}\int_{|z|=r<1} \frac{f(z)}{z^{n+1}(1-z)^k}\,dz \\ &= \frac{1}{2\pi r^n}\int_0^{2\pi} \frac{f(re^{i\theta})}{e^{in\theta}(1-re^{i\theta})^k}\,d\theta. \end{aligned}$$

Hence

$$\begin{aligned} |S_n(k)| &\leqq \frac{1}{2\pi r^n}\int_0^{2\pi} \frac{|f(re^{i\theta})|}{|1-re^{i\theta}|^k}\,d\theta \\ &\leqq \max_{0\leqq\theta<2\pi} \frac{|f(re^{i\theta})|}{2\pi r^n}\int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^k}. \end{aligned}$$

By the well known "Distortion Theorem"

$$\max_{0\leqq\theta<2\pi} |f(re^{i\theta})| \leqq \frac{r}{(1-r)^2} \qquad (r<1).$$

To estimate the integral expression we write

$$|1-re^{i\theta}| = (1-2r\cos\theta+r^2)^{1/2},$$

so that [2]

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^k} &= \frac{1}{2\pi(1-r^2)^{k/2}}\int_0^{2\pi} \frac{(1-r^2)^{k/2}}{(1-2r\cos\theta+r^2)^{k/2}}\,d\theta \\ &= (1-r^2)^{-k/2}P_{k/2-1}\left\{\frac{1+r^2}{1-r^2}\right\}. \end{aligned}$$

Here $P_{n(x)}$ is the Legendre function of the first kind of order n. Since $\lim_{x\to\infty} P_n(x)/x^n = 2^{-n}\Gamma(2n+1)/\Gamma^2(n+1)$ [3, p. 62], we may write

$$|S_n(k)| \leqq r^{-n+1}(1-r)^{-2}(1-r^2)^{-k/2}\left(\frac{1+r^2}{1-r^2}\right)^{k/2-1}\phi_k(r),$$

where $\lim_{r\to 1} \phi_k(r) = 2^{-k/2+1}\Gamma(k-1)/\Gamma^2(k/2)$.

Thus far r has been any number between 0 and 1. We now specify $r=1-(k+1)/n$. Then

$$|\sigma_n(k)| \leqq \left(1-\frac{k+1}{n}\right)^{-n+1}\left(\frac{k+1}{n}\right)^{-k-1}(1+r)^{-k/2}$$
$$\cdot\left(\frac{1+r^2}{1+r}\right)^{k/2-1}\phi_k(r)(C_{n+k,k+1})^{-1}.$$

Since $C_{n+k,k+1}\cong n^{k+1}/\Gamma(k+2)$, we readily compute $\limsup_{n\to\infty}\sigma_n(k) \leqq A(k)$, where

$$A(k)=\frac{e^{k+1}\Gamma(k+2)\Gamma(k-1)}{(k+1)^{k+1}2^{k-1}\Gamma^2(k/2)}.$$

That $\lim_{k\to\infty}A(k)=1$ may now be verified by using Stirling's formula for $\Gamma(k)$.

While the numbers $A(k)$ do tend to unity they decrease very slowly. Computations yield $A(2)=2.23$, $A(4)=1.42$, $A(6)=1.26$, $A(10)=1.15$, $A(20)=1.07$. Hence even $A(4)$ is greater than $e/2$. A better estimate of $\limsup_{n\to\infty}\sigma_n(k)$ for small k can be obtained by use of the following lemma, recently announced by Bazilevič [1].

LEMMA. *The intersection of the circumference* $|w|=x$, $x\geqq re^{\pi/e}$, *with the domain* $D(r)$ *on which* $f(z)$ *maps* $|z|\leqq r<1$ *has linear measure not greater than that of the intersection of the same circumference with the domain* $D^*(r)$ *on which* $f^*(z)=z/(1-z)^2$ *maps* $|z|\leqq r$.

It follows at once from the lemma that the area $\psi(r)$ of the region $D(r)$ is not greater than $\pi r^2e^{2\pi/e}$ plus the area $\psi^*(r)$ of $D^*(r)$. Further

$$\begin{aligned}\psi^*(r) &= \int_0^{2\pi}d\theta\int_0^r r\,|f^{*\prime}(re^{i\theta})|^2dr \\ &= \pi\sum_{j=1}^{\infty}j^3r^{2j} \qquad (2) \\ &= \frac{\pi r^2(1+4r^2+r^4)}{(1-r^2)^4}.\end{aligned}$$

We may now prove our second theorem.

THEOREM 2. *Let* $k\geqq 1$. *Then*

$$\limsup_{n\to\infty}\sigma_n(k) \leqq \frac{ke^{k+1}\Gamma^{1/2}(2k-1)}{(k+1)^k2^{k+1/2}} = B(k). \qquad (3)$$

In particular,

$$B(1)=1.307,\ B(2)=1.116,\ B(3)=1.109.$$

PROOF. We apply Schwarz's inequality to (1) to get

$$(4)\qquad |S_n(k)|^2 \leqq r^{-2n}\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\right\}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|^{2k}}\right\}.$$

To estimate the first integral we write

$$\begin{aligned} I &= \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \\ &= \sum_{j=1}^{\infty} |a_j|^2 r^{2j} \\ &= 2\int_0^r \sum_{j=1}^{\infty} j\,|a_j|^2 r^{2j-1} dr \\ &= \frac{2}{\pi}\int_0^r \frac{\psi(r)}{r}\,dr. \end{aligned}$$

Hence (2) yields

$$I \leqq \frac{2}{\pi}\int_0^r \left\{r\pi e^{2\pi/e} + \frac{\pi r(1+4r^2+r^4)}{(1-r^2)^4}\right\} dr.$$

An integration by parts then gives

$$I \leqq \frac{2}{(1-r^2)^3} + \frac{g(r)}{(1-r)^2},$$

where $g(r)$ is a function bounded for $0 \leqq r \leqq 1$.

The second integral of (4) can be handled as in Theorem 1. Thus

$$|S_n(k)|^2 \leqq r^{-2n}\left\{\frac{2}{(1-r^2)^3} + \frac{g(r)}{(1-r)^2}\right\}(1-r^2)^{-k}P_{k-1}\left\{\frac{1+r^2}{1-r^2}\right\}.$$

On choosing

$$r = 1 - \frac{k+1}{n}$$

and carrying out the computations as before, we get the results asserted.

It is interesting to note that for values of $k>3$ the numbers $B(k)$

defined in (3) increase, behaving like $2^{-1}\{\pi k\}^{1/4}$ for large k. The technique of using the Schwarz inequality is thus ineffective for the study of $\lim\sup_{n\to\infty} \sigma_n(k)$ for all but the smaller values of k.

Bibliography

1. I. E. Bazilevič, *On distortion theorems and the coefficients of univalent functions*, Doklady Akademii Nauk SSSR (N.S.) vol. 65 (1949) pp. 253–258 (Russian).

2. E. Pinney, *On a note of Galbraith and Green*, Bull. Amer. Math. Soc. vol. 54 (1948) p. 527.

3. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939.

University of California, Los Angeles