A NOTE ON ERGODIC THEORY

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The purpose of this paper is to present solutions to certain problems in ergodic theory suggested by Einar Hille in his book *Functional analysis and semi-groups* [1].[1]

Let $T(\xi)$ $(\xi>0)$ be a semi-group of linear bounded transformations on a complex Banach space $\mathfrak{X}$ to itself. Following Hille, we say that $T(\xi)$ is ergodic at infinity (or at 0) if it has a generalized limit of some sort when $\xi\to\infty$ (or $0+$). We define the Abel mean

$$L[\lambda, T(\cdot)x] = \lambda \int_0^\infty \exp(-\lambda\xi)T(\xi)x d\xi \tag{1}$$

and the Cesaro-α (C-α) mean

$$a[\xi, T(\cdot)x, \alpha] = \alpha\xi^{-\alpha} \int_0^\xi (\xi - \tau)^{\alpha-1}T(\tau)x d\tau. \tag{2}$$

In order that the integrals involved shall have a sense, we impose the following restrictions on $T(\xi)$:

(i) $T(\xi)$ is strongly measurable for $\xi > 0$,

(ii$_s$) $\int_0^\infty \exp(-\lambda\xi)\|T(\xi)x\|d\xi$ exists for $\lambda > 0$ and all $x \in \mathfrak{X}$

or

(ii$_u$) $\int_0^\infty \exp(-\lambda\xi)\|T(\xi)\|d\xi$ exists for $\lambda > 0$.

In theorems referring to limiting processes at infinity we shall also impose the restriction

(iii) $\lim_{\eta\to 0+} \eta^{-1} \int_0^\eta T(\xi)x d\xi = x$ for all $x \in \mathfrak{X}$.

Condition (i) alone implies that $T(\xi)$ is strongly continuous for $\xi>0$ (see [1, Theorem 9.2.1] and [2]); whereas (ii$_s$) implies the existence of the Abel mean for $\lambda>0$. If (ii$_u$) is satisfied, the infinitesimal generator A of the semi-group operator will be a closed linear trans-

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[1] Numbers in brackets refer to the references cited at the end of the paper.

formation. If conditions (i), (ii_s), and (iii) are all satisfied, then the resolvent theory (see [1, sec. 11.8]) is valid; in particular the linear bounded operator

$$(3)\qquad R(\lambda; A)x = \int_0^\infty \exp(-\lambda\xi)T(\xi)x d\xi$$

exists for Re $(\lambda) > 0$, satisfies the first resolvent equation, and

$$(4)\qquad \begin{aligned} (\lambda I - A)R(\lambda; A)x &= x \qquad \text{for } x \in \mathfrak{X}, \\ R(\lambda; A)(\lambda I - A)x &= x \qquad \text{for } x \in \mathfrak{D}[A], \end{aligned}$$

where $\mathfrak{D}[A]$, the domain of A, is now dense in $\mathfrak{X}$. It will be shown in Example 2 that conditions (i), (ii_u), and (iii) do not imply the boundedness of $\|T(\xi)\|$ in a neighborhood of the origin. Finally, if U is a linear operator on $\mathfrak{X}$ to itself, we denote by $\mathfrak{R}[U]$ the range of U and by $\mathfrak{Z}[U]$ the zeros of U.

Our principal result is the proof of the following theorem conjectured by Hille [1, p. 301].

THEOREM 1. *If the semi-group operator $T(\xi)$ satisfies conditions* (i), (ii_u), (iii); *and if in addition $\mathfrak{R}[A^2]$ is closed and $\lim_{\lambda\to 0+} \lambda^2 R(\lambda; A)x = \theta$ for all $x \in \mathfrak{X}$, then $T(\xi)$ is Abel ergodic at infinity in the uniform topology.*

We first show that $\mathfrak{D}[A^2]$ is dense in $\mathfrak{X}$. In fact, for any $x \in \mathfrak{X}$, set

$$(5)\qquad \begin{aligned} y(\tau) &= \frac{6}{\tau^3}\int_0^\tau (\tau - \xi)\xi T(\xi)x d\xi \\ &= \frac{6}{\tau^3}\int_0^\tau (2s - \tau)s\left(s^{-1}\int_0^s T(\xi)x d\xi\right) ds. \end{aligned}$$

It follows from (iii) that $y(\tau) \to x$ as $\tau \to 0+$. Hence it will be sufficient to show that $y(\tau) \in \mathfrak{D}[A^2]$. However a straightforward calculation (compare with [1, Theorem 11.5.1]) shows that $Ay(\tau) = (6/\tau^3)\int_0^\tau(\tau - 2\xi)T(\xi)x d\xi$ and $A^2 y(\tau) = (6/\tau^3)[2\int_0^\tau T(\xi)x d\xi - \tau T(\tau)x - \tau x]$. Next, if $x \in \mathfrak{D}[A^2]$, we deduce from (4) that

$$(6)\qquad \lambda^2 R(\lambda; A)x - \lambda x - Ax = R(\lambda; A)A^2 x = A^2 R(\lambda; A)x.$$

By hypotheses, $\lim_{\lambda\to 0+} \lambda^2 R(\lambda; A)x = \theta$, so that $\lim_{\lambda\to 0+} A^2 R(\lambda; A)x = -Ax$. Thus if $x \in \mathfrak{D}[A^2]$, then $Ax \in \mathfrak{R}[A^2]$ (closed by hypothesis). In other words $\mathfrak{R}[A^2] \supset \mathfrak{D}[A] \cap \mathfrak{R}[A]$, from which it follows that $\mathfrak{D}[A] \cap \mathfrak{R}[A] = \mathfrak{D}[A] \cap \mathfrak{R}[A^2]$. Further, A is one-to-one on $\mathfrak{D}[A] \cap \mathfrak{R}[A^2]$. For, if $x \in \mathfrak{D}[A] \cap \mathfrak{R}[A]$ and $Ax = \theta$, then there exists a $y \in \mathfrak{D}[A^2]$ such that $x = Ay$ and by (6)

$$(7)\qquad x = Ay = \lim_{\lambda\to 0+} R(\lambda; A)A^2y = \theta.$$

We define A_1 to be the contraction of A on $\mathfrak{R}[A^2]$. Then A_1 is closed, one-to-one, and $\mathfrak{R}[A_1]=\mathfrak{R}[A^2]$. It follows that A_1^{-1} is a linear bounded operator [1, Theorem 2.13.9] on $\mathfrak{R}[A^2]$ to itself. For $y \in\mathfrak{R}[A^2]$, set $x=A_1^{-1}y$. Then $x\in\mathfrak{D}[A_1]$ and by (4)

$$R(\lambda; A)y = -x + \lambda R(\lambda; A)x.$$

Since $\lambda^2R(\lambda; A)x\to\theta$ as $\lambda\to 0+$ by hypothesis, $\lim_{\lambda\to 0+} \lambda R(\lambda; A)y=\theta$ for all $y\in\mathfrak{R}[A^2]$. By a generalized version of the Banach-Steinhaus theorem [1, Theorem 2.12.2] there exists a constant M such that $\|\lambda R(\lambda; A)y\| \leqq M\|y\|$ for $0<\lambda\leqq 1$ and all $y\in\mathfrak{R}[A^2]$. Hence

$$(8)\qquad \|\lambda R(\lambda; A)y\| \leqq \lambda\|A_1^{-1}\| \mid 1 + M \mid \|y\|$$

for all $y\in\mathfrak{R}[A^2]$ and $0<\lambda\leqq 1$.

$\mathfrak{Z}[A]$ can be characterized as the set of all x such that $T(\xi)x\equiv x$. In fact, if $x\in\mathfrak{Z}[A]$,

$$T(\xi)x - x = \int_0^\xi \frac{dT(\tau)x}{d\tau}d\tau = \int_0^\xi T(\tau)Axd\tau = \theta;$$

the converse is obvious. It follows that $\mathfrak{Z}[A]$ is a closed linear subspace and that $\lambda R(\lambda; A)x\equiv x$ for each $x\in\mathfrak{Z}[A]$. Thus $\mathfrak{Z}[A]\cap\mathfrak{R}[A^2] =\phi$. On the other hand $(\mathfrak{Z}[A]\oplus\mathfrak{R}[A^2])^-=\mathfrak{X}$.[2] For if $x\in\mathfrak{D}[A^2]$, then $A^2x=z\in\mathfrak{R}[A^2]$. Let $x_1=A_1^{-2}z$; then $x_1\in\mathfrak{D}[A^2]\cap\mathfrak{R}[A^2]$. Hence $A^2(x-x_1)=0$. Applying (7) (with $y=x-x_1$), we see that $x_2=x-x_1 \in\mathfrak{Z}[A]$. In other words $x=x_1+x_2\in\mathfrak{R}[A^2]\oplus\mathfrak{Z}[A]$. The conclusion then follows from the fact that $(\mathfrak{D}[A^2])^-=\mathfrak{X}$. We can, however, prove much more; in fact, $\mathfrak{X}=\mathfrak{R}[A^2]\oplus\mathfrak{Z}[A]$. Given $x\in\mathfrak{X}$, as we have just shown, there exists a sequence $x_n=y_n+z_n$ such that $x_n\to x$, $y_n\in\mathfrak{R}[A^2]$, and $z_n\in\mathfrak{Z}[A]$. Now

$$[T(\xi) - I]x_n = [T(\xi) - I]y_n = [T(\xi)- I]A^2w_n = A^2[T(\xi) - I]w_n$$

where $w_n\in\mathfrak{D}[A^2]$. Hence $[T(\xi)-I]x\in\mathfrak{R}[A^2]$ for all $\xi>0$. Since $\mathfrak{R}[A^2]$ is closed and linear

$$\frac{6}{\tau^3}\int_0^\tau (\tau - \xi)\xi(T(\xi) - I)xd\xi \in \mathfrak{R}[A^2].$$

However, as we have seen in (5), $y(\tau)\in\mathfrak{D}[A^2]$ and as above $\mathfrak{D}[A^2]$

[2] $(\mathfrak{M})^-$ denotes the closure of the set $\mathfrak{M}$.

$\subset \mathfrak{N}[A^2] \oplus \mathfrak{Z}[A]$. Therefore

$$x = \frac{6}{\tau^3}\int_0^\tau (\tau - \xi)\xi T(\xi) x d\xi - \frac{6}{\tau^3}\int_0^\tau (\tau - \xi)\xi [T(\xi) - I] x d\xi$$

belongs to $\mathfrak{N}[A^2] \oplus \mathfrak{Z}[A]$. Thus for each $x \in \mathfrak{X}$ there exists a unique decomposition $x = y + z$ where $y \in \mathfrak{N}[A^2]$ and $z \in \mathfrak{Z}[A]$. We define the projection operator $Px = z$. Since both $\mathfrak{N}[A^2]$ and $\mathfrak{Z}[A]$ are closed, P will be closed and since it is defined for all $x \in \mathfrak{X}$ it will be bounded [**1**, Theorem 2.13.9]. Now $Px \in \mathfrak{Z}[A]$, so that $\lambda R(\lambda; A)Px = Px$. Hence by (8)

$$\|\lambda R(\lambda; A)x - Px\| = \|\lambda R(\lambda; A)(1 - P)x\| \leqq \lambda K\|x\|$$

and $T(\xi)$ is therefore uniformly ergodic at infinity to the projection operator P.

We remark that the converse of Theorem 1 follows directly from [**1**, Theorem 14.8.3 and 14.8.4].

THEOREM 2. *If the semi-group operator* $T(\xi)$ *satisfies conditions* (i) *and* (ii$_s$), *if* $T(\xi)$ *is strongly Abel ergodic at* 0, *and if* $\|a(\xi, T(\cdot)x, \alpha)\| \leqq M\|x\|$ *for* $0 < \xi < 1$, *then* $T(\xi)$ *is strongly* (C-α) *ergodic at* 0.

Condition (ii$_s$) can be replaced by the weaker condition $\int_0^1 \|T(\xi)x\| d\xi < \infty$ for all $x \in \mathfrak{X}$. It is interesting to note that Theorem 2 does not have its counterpart in the infinite limit. In the latter case, Hille [**1**, Theorem 14.7.2] has proved a much weaker theorem and this with the help of a Tauberian theorem, whereas the following proof is of an elementary nature.

By assumption, $\lim_{\lambda \to \infty} L[\lambda, T(\cdot)x] = Jx$ for all $x \in \mathfrak{X}$. As Hille has shown [**1**, Theorem 14.6.2], J is a bounded projection operator, $X = \mathfrak{N}[J] \oplus \mathfrak{Z}[J]$, $\mathfrak{Z}[J] = \bigcap_\xi \mathfrak{Z}[T(\xi)]$, and $\mathfrak{N}[J] = (\mathfrak{D}[A])^-$. Hence for any $x \in \mathfrak{X}$, $x = Jx + (I - J)x = y + z$. Since $y \in \mathfrak{N}[J] = (\mathfrak{D}[A])^-$, there exists a sequence $y_n \in \mathfrak{D}[A]$ such that $y_n \to y$. For $y_n \in \mathfrak{D}[A]$, $\lim_{\xi \to 0+} T(\xi)y_n = y_n$; whereas for $z \in \mathfrak{Z}[J]$, $T(\xi)z \equiv 0$. Hence

$$\lim_{\xi \to 0+} a[\xi, T(\cdot)(y_n + z), \alpha] = y_n.$$

By hypothesis $a[\xi, T(\cdot)x, \alpha]$ is a family of linear continuous operators, uniformly bounded for $\xi \in (0, 1)$. Hence the iterated limits exist and their order may be interchanged; that is,

$$\lim_{\xi \to 0+} a[\xi, T(\cdot)x, \alpha) = \lim_{\xi \to 0+} \lim_n a[\xi, T(\cdot)(y_n + z), \alpha]$$

$$= \lim_n y_n = y = J(x).$$

For purposes of the following theorem, we define $\mathfrak{X}_1=(\mathfrak{R}[A]\oplus\mathfrak{Z}[A])^-$, A_1 to be the retraction of A on $\mathfrak{X}_1$, and A_2 to be the retraction of A on $(\mathfrak{R}[A])^-$. We then have the following theorem.

THEOREM 3. *If the semi-group operator* $T(\xi)$ *satisfies conditions* (i), (ii$_s$), (iii), *and if* $\|\lambda R(\lambda;\ A)\|\leqq M$ *for* $0<\lambda<1$, *then* $\lim_{\lambda\to 0+}\lambda R(\lambda,\ A)x$ *exists for all* $x\in\mathfrak{X}_1$, $T(\xi)\mathfrak{X}_1\subset\mathfrak{X}_1$, *and* $(\mathfrak{R}[A])^-=(\mathfrak{R}[A_1])^-=(\mathfrak{R}[A_2])^-$.

As in Theorem 1, $\mathfrak{Z}[A]$ is closed and $\lambda R(\lambda;\ A)x\equiv x$ if $x\in\mathfrak{Z}[A]$. If $x\in\mathfrak{R}[A]$, then $x=Ay$ and by (4), $\lambda R(\lambda;\ A)x=\lambda^2R(\lambda;\ A)y-\lambda y$ so that $\lambda R(\lambda;\ A)x\to\theta$ as $\lambda\to 0+$. Thus this limit exists for all $x\in\mathfrak{R}[A]\oplus\mathfrak{Z}[A]$ and since $\|\lambda R(\lambda;\ A)\|\leqq M$ for $\lambda\in(0,\ 1)$, it will exist for all $x\in\mathfrak{X}_1$. Now if $x=y+z$ where $y\in\mathfrak{R}[A]$ and $z\in\mathfrak{Z}[A]$, then $T(\xi)z\equiv z$ and $T(\xi)y=T(\xi)Aw=AT(\xi)w$ for some $w\in\mathfrak{D}[A]$. Hence $T(\xi)x=z+AT(\xi)w$ belongs to $\mathfrak{X}_1$, and since $T(\xi)$ is continuous the same applies to all $x\in\mathfrak{X}_1$. Therefore relative to $\mathfrak{X}_1$, $T(\xi)$ is strongly Abel ergodic at infinity. As Hille [1, Theorem 14.6.1] has shown, there exists a bounded projection operator P_1 on $\mathfrak{X}_1$ such that $\lim_{\lambda\to 0+}L[\lambda,\ T(\cdot)x]=P_1x$ for all $x\in\mathfrak{X}_1$, $\mathfrak{Z}[P_1]=(\mathfrak{R}[A_1])^-$, $\mathfrak{R}[P_1]=\mathfrak{Z}[A_1]$, and $\mathfrak{X}_1=(\mathfrak{R}[A_1])^-\oplus\mathfrak{Z}[A_1]$. Now by the definition of $\mathfrak{X}_1$, it follows that $\mathfrak{Z}[A]=\mathfrak{Z}[A_1]$. Clearly $(\mathfrak{R}[A])^-\supset(\mathfrak{R}[A_1])^-$. The converse is likewise true. For if $x\in\mathfrak{R}[A]$, then as above $\lambda R(\lambda;\ A)x\to\theta$. The same is true of $x\in(\mathfrak{R}[A])^-$ since $\|\lambda R(\lambda;\ A)\|\leqq M$ for $0<\lambda<1$. Hence if $x\in(\mathfrak{R}[A])^-$, then $x\in\mathfrak{Z}[P_1]=(\mathfrak{R}[A_1])^-$. Finally if $y\in\mathfrak{R}[A_1]$, then there exists an $x\in\mathfrak{D}[A_1]$ such that $y=A_1x$. By the above decomposition, $P_1x\in\mathfrak{Z}[A_1]$; hence $w=(I-P_1)x\in\mathfrak{D}[A_1]\cap(\mathfrak{R}[A_1])^-$ and $A_1w=y$. In other words $\mathfrak{R}[A_2]=\mathfrak{R}[A_1]$.

COROLLARY 1. *If* $T(\xi)$ *satisfies the hypothesis of Theorem* 3 *and if* $\mathfrak{X}=(\mathfrak{R}[A]\oplus\mathfrak{Z}[A])^-$, *then* $T(\xi)$ *is strongly Abel ergodic at infinity.*

This result is slightly stronger than that obtained by Hille [1, Theorem 14.7.1 (2)].

COROLLARY 2. *If* $T(\xi)$ *satisfies the hypothesis of Theorem* 3, *then* $\lambda=0$ *is either in the resolvent set or in the continuous spectrum of* A_2.

Clearly $(\mathfrak{R}[A])^-=(\mathfrak{R}[A_2])^-$ rules out the residual spectrum and $\mathfrak{Z}[A]\cap(\mathfrak{R}[A])^-=\phi$ rules out the point spectrum.

Hille has conjectured [1, p. 295] that a semi-group operator satisfying the hypothesis of Theorem 3 and such that $\lambda=0$ does not belong to the residual spectrum of either A or A_2 would necessarily be strongly Abel ergodic at infinity. It is clear from Corollary 2 that the condition imposed on A_2 is already implied by the other restrictions. Thus in order to construct a counter-example to this conjecture it will be sufficient to produce a non-Abel-ergodic semi-group satisfying the hypothesis of Theorem 3 and having $\lambda=0$ in the point spectrum of A. This we do in the following example.

EXAMPLE 1. Let $\mathfrak{X}=m$, the space of bounded sequences $\{a_n\}$, with norm $\|\{a_n\}\|=\text{LUB }|a_n|$. We define $T(\xi)\{a_n\}=\{a'_n\}$ by

$$a'_0 = a_0,$$
$$a'_n = a_n \exp(-i\xi/n) \qquad \text{for } n \geqq 1.$$

Clearly $\|T(\xi)\|=1$, and

$$\|(T(\xi)-I)\{a_n\}\| \leqq \|\{a_n\}\| \cdot |\exp(i\xi)-1|.$$

Thus $T(\xi)$ is uniformly continuous at the origin and hence A is a linear bounded operator. In fact for $A\{a_n\}=\{a'_n\}$, we have

$$a'_0 = 0,$$
$$a'_n = ia_n/n \qquad \text{for } n \geqq 1.$$

It is clear that $\lambda=0$ is in the point spectrum of A. Finally if we have $\lambda R(\lambda; A)\{a_n\}=\{a'_n\}$, then

$$a'_0 = a_0,$$
$$a'_n = a_n n\lambda/(n\lambda+i) \qquad \text{for } n \geqq 1,$$

and since $|n\lambda/(n\lambda+i)| \leqq 1$ for $\lambda>0$, it follows that $\|\lambda R(\lambda; A)\| \leqq 1$ for $\lambda>0$. Now if $T(\xi)$ were strongly ergodic at infinity to the projection operator P, then

$$\mathfrak{R}[P] = \mathfrak{Z}[A] = [(b, 0, 0, 0, \cdots)].$$

However this limit fails for $\{a_n \equiv 1\}$ since

$$\|(\lambda R(\lambda; A) - P)(1, 1, \cdots)\| = \|(0, \cdots, n\lambda/(n\lambda+i), \cdots)\| = 1$$

for all $\lambda>0$.

EXAMPLE 2. We next define a semi-group operator $T(\xi)$ satisfying conditions (i), (ii$_u$), (iii) and such that $\lim\sup_{\xi\to 0+}\|T(\xi)\|=\infty$. We start with a sequence of two-dimensional normed linear spaces $\mathfrak{X}_n$ and define $\mathfrak{X}$ to be the set of all sequences $\{x_n \in \mathfrak{X}_n\}$ such that $\sum\|x_n\|<\infty$, with norm $\|\{x_n\}\|=\sum\|x_n\|$. $\mathfrak{X}_n$ itself is defined as the set of all complex-valued pairs $x_n=(y, z)$ with norm $\|x_n\|=(|y|^2+n|z|^2)^{1/2}$. We now define a semigroup operator $T_n(\xi)x_n=x'_n$ on $\mathfrak{X}_n$ to $\mathfrak{X}_n$ such that

$$y' = \exp[-(n+in^3)\xi](y\cos n\xi - z\sin n\xi),$$
$$z' = \exp[-(n+in^3)\xi](y\sin n\xi + z\cos n\xi).$$

It is clear that $\|T_n(\xi)\| \leqq n^{1/2}\exp(-n\xi)$ and that $\|T_n(\pi/2n)\|$

$=n^{1/2}\exp(-\pi/2)$ [as can be seen by operating on (1, 0)]. The semi-group operator $T(\xi)$ is defined by $T(\xi)\{x_n\}=\{T_n(\xi)x_n\}$. It follows from the way the norm in X has been defined that $\|T(\xi)\|$ $=\text{LUB}\ \|T_n(\xi)\|\leqq(2e\xi)^{-1/2}$. Now $T_n(\xi)x_n$ is clearly continuous in ξ, and since the $\lim_k \sum_k^\infty\|T_n(\xi)x_n\|=0$ uniformly in $\xi\geqq\delta>0$, $T(\xi)\{x_n\}$ will itself be strongly continuous for $\xi>0$. In this case $\|T(\xi)\|$ is measurable and, from the above upper bound, we have

$$\int_0^\infty \exp(-\lambda\xi)\|T(\xi)\|d\xi \leqq \int_0^\infty \exp(-\lambda\xi)(2e\xi)^{-1/2}d\xi < \infty \qquad \text{for } \lambda > 0.$$

Further, since $\|T(\pi/2n)\|\geqq n^{1/2}\exp(-\pi/2)$, $\lim\sup_{\xi\to 0+}\|T(\xi)\|=\infty$. It remains to show that (iii) is satisfied. We define

$$S(\eta)x = \eta^{-1}\int_0^\eta T(\xi)xd\xi = \{S_n(\eta)x_n\}.$$

Here

$$S_n(\eta)x_n = \eta^{-1}\int_0^\eta T_n(\xi)x_nd\xi = x_n' = (\alpha(\eta)y - \beta(\eta)z,\ \beta(\eta)y + \alpha(\eta)z)$$

where

$$\alpha(\eta) = \eta^{-1}\int_0^\eta \exp[-(n+in^3)\xi]\cos n\xi d\xi,$$

$$\beta(\eta) = \eta^{-1}\int_0^\eta \exp[-(n+in^3)\xi]\sin n\xi d\xi.$$

A straightforward calculation shows that $|\alpha(\eta)|\leqq 1$ and $|\beta(\eta)|$ $\leqq 2/n^2$. Hence for $|y|^2+n|z|^2=1$,

$$\begin{aligned}|y'|^2 + n|z'|^2 \leqq |y|^2 + n|z|^2 + (4/n)|y||z| + (4/n^2)|y||z| \\ + (4/n^2)|y|^2 + (4/n^4)|z|^2 \\ \leqq 1 + 16/n \leqq 17.\end{aligned}$$

Therefore $\|S(\eta)\|\leqq\text{LUB}\ \|S_n(\eta)\|\leqq 17$. For ultimately zero vectors, it is clear that $T(\xi)x\to x$ as $\xi\to 0+$ and hence that $S(\eta)x\to x$ as $\eta\to 0+$. Since such vectors are dense in $\mathfrak{X}$ and since $\|S(\eta)\|\leqq 17$, it follows that $S(\eta)x\to x$ as $\eta\to 0+$ for all $x\in\mathfrak{X}$.

EXAMPLE 3. We conclude with an example of a strongly continuous group operator $T(\xi)$ on $(-\infty,\infty)$ for which $\|T(\xi)\|$ is not continuous. This example settles a question raised by Hille [1, p. 184]. As in Example 2, $\mathfrak{X}=\prod\mathfrak{X}_n$ and $\|x\|=\sum\|x_n\|$. In this case, however, $\|x_n\|=|y|$

$+|z|$, and $T_n(\xi)x_n = x'_n$ is defined by

$$y' = y \cos n\xi - z \sin n\xi, \qquad z' = y \sin n\xi + z \cos n\xi,$$

which is simply a rotation in $\mathfrak{X}_n$. Since the unit sphere in $\mathfrak{X}_n$ is a square, the maximum expansion will be $2^{1/2}$. Hence $\|T(\xi)\| = \text{LUB } \|T_n(\xi)\| \leqq 2^{1/2}$. For $\xi = k\pi$ $(k = 0, \pm 1, \pm 2, \cdots)$, $\|T(\xi)x\| = \|x\|$ so that $\|T(k\pi)\| = 1$. For $n\xi = (nk \pm 1/4)\pi$, set $x_n = (1, 0)$ and $x_j = \theta$ for $j \neq n$. Then $x'_n = (\pm 1/2^{1/2}, \pm 1/2^{1/2})$ so that $\|T[(k \pm 1/4n)\pi]\| = 2^{1/2}$. Therefore $\|T(\xi)\|$ is discontinuous at the points $k\pi$. One can show as in Example 2 that $T(\xi)x$ is strongly continuous on $(-\infty, \infty)$.

References

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