NOTE ON A THEOREM OF GELFAND AND ŠILOV

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The purpose of this note is to give a simplified proof of a theorem of Gelfand and Šilov in the theory of normed, commutative rings.

Let K be a complex Banach space which is also a commutative ring with unit element e, the norm being subject to the conditions $\|e\| = 1$ and $\|xy\| \leqq \|x\| \, \|y\|$ for all x and y in K. Let $\mathfrak{M}$ be the space of maximal ideals of K. Then, for each element $x \in K$, and each maximal ideal $M \in \mathfrak{M}$, there is a unique complex number $x(M)$ defined by

$$x \equiv x(M)e \pmod{M}$$

and having the following properties:

$$e(M) = 1, \tag{1}$$

$$(x + y)(M) = x(M) + y(M), \tag{2}$$

$$(xy)(M) = x(M)y(M), \tag{3}$$

$$(\alpha x)(M) = \alpha x(M), \tag{4}$$

$$|x(M)| \leqq \|x\|, \tag{5}$$

for all x and y in K and any complex number α [1].[1]

Gelfand [1] introduces a topology in $\mathfrak{M}$ by defining a neighborhood U of M_0 as follows:

$$U = \{M; \, | x_i(M) - x_i(M_0) | < \alpha_i; x_i \in K; \alpha_i > 0; i = 1, 2, \cdots, k\};$$

and he proves that in this topology, $\mathfrak{M}$ is a compact Hausdorff space, and that this is the unique topology in which all functions $x(M)$, $x \in K$, are continuous and $\mathfrak{M}$ is compact.

If the ring K also has the property that, for every $x \in K$, there exists an $x^* \in K$ such that $x(M)$ and $x^*(M)$ are complex conjugates for all M, then the functions $x(M)$ are dense in the set of all continuous functions on $\mathfrak{M}$. This result is proved by Gelfand and Šilov [2] by a method depending on two other topologies in the space $\mathfrak{M}$. We give here a simpler and more direct proof of this theorem, making use of only the one topology defined above.

LEMMA. *If F_1 and F_2 are any two disjoint closed sets in $\mathfrak{M}$, and $0 < \epsilon_1 < 1$, $0 < \epsilon_2 < 1$, there exists an $x \in K$ such that*

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[1] Numbers in brackets refer to the references at the end of the paper.

$$0 \leqq x(M) \leqq 1, \qquad M \in \mathfrak{M},$$
$$0 \leqq x(M) \leqq \epsilon_1, \qquad M \in F_1,$$
$$1 - \epsilon_2 \leqq x(M) \leqq 1, \qquad M \in F_2.$$

PROOF. Let $M_0 \in F_1$. Let $U = \{M; \ |x_i(M) - x_i(M_0)| < \alpha; \ i = 1, 2, \cdots k\}$ be a neighborhood of M_0 which does not intersect F_2. Let $x_i' = x_i - x_i(M_0)e$, $y' = \sum_{i=1}^{k} x_i'(x_i')^*$. Then $y'(M)$ is non-negative for all M, is zero for $M = M_0$, and has a positive lower bound α^2 for $M \in F_2$. Let $y = y'/\|y'\|$, $2\delta = \alpha^2/\|y'\| \leqq 1$; then

$$0 \leqq y(M) \leqq 1, \qquad M \in \mathfrak{M},$$
$$y(M_0) = 0,$$
$$0 < 2\delta \leqq y(M) \leqq 1, \qquad M \in F_2.$$

With each $M_0 \in F_1$ we associate a y with the properties just given and a neighborhood of M_0: $U = \{M; \ y(M) < \delta\}$. From these neighborhoods we select a finite covering of F_1, denoted by $U_1, U_2, \cdots, U_N$, and we let the y and δ associated with U_k be denoted by y_k and δ_k. Now let $z_k = e - (e - y_k^n)^m$, where n and m are positive integers. We then have

$$0 \leqq z_k(M) \leqq 1, \qquad M \in \mathfrak{M},$$
$$0 \leqq z_k(M) < 1 - (1 - \delta_k^n)^m, \qquad M \in U_k,$$
$$1 - (1 - 2^n\delta_k^n)^m \leqq z_k(M) \leqq 1, \qquad M \in F_2.$$

If, for each value of n, we now choose m as the integer nearest $(2/3\delta_k)^n$, then, as $n \to \infty$, $m \log (1 - \delta_k^n) \to 0$ and $m|\log (1 - 2^n\delta_k^n)| \to \infty$; hence $(1 - \delta_k^n)^m \to 1$ and $(1 - 2^n\delta_k^n)^m \to 0$. This follows from the inequality $h < |\log (1 - h)| < 2h$ for $0 < h \leqq 1/2$. We can therefore choose n so large that

$$0 \leqq z_k(M) < \epsilon_1, \qquad M \in U_k,$$
$$(1 - \epsilon_2)^{1/N} \leqq z_k(M) \leqq 1, \qquad M \in F_2.$$

The function $x = z_1 z_2 \cdots z_N$ will then satisfy the conditions of the lemma.

THEOREM. *For any complex-valued function $f(M)$, continuous on $\mathfrak{M}$, and any $\epsilon > 0$, there exists an $x \in K$ such that $|f(M) - x(M)| < \epsilon$ for all M.*

PROOF. We first prove the theorem for a real-valued continuous function $f(M)$. Let K_R be the set $\{x\}$ where $x \in K$ and $x(M)$ is real-valued, and let

$$a = \inf_{x \in K_R} \sup_{M \in \mathfrak{M}} |f(M) - x(M)|.$$

Let us assume $a > 0$. Then there exists an $x_0 \in K_R$ such that

$$\sup_{M \in \mathfrak{M}} |f(M) - x_0(M)| < 1.1a.$$

Let $g(M) = f(M) - x_0(M)$, and let

$$\begin{aligned} F_1 &= \{M; g(M) \geqq .5a\}, \\ G &= \{M; |g(M)| < .5a\}, \\ F_2 &= \{M; g(M) \leqq -.5a\}. \end{aligned}$$

By the preceding lemma we can find $x_1 \in K_R$, $x_2 \in K_R$ such that

$$\begin{array}{lll} 0 \leqq x_1(M) \leqq .4a, & 0 \leqq x_2(M) \leqq .4a, & M \in \mathfrak{M}, \\ 0 \leqq x_1(M) \leqq .1a, & .3a \leqq x_2(M) \leqq .4a, & M \in F_2, \\ .3a \leqq x_1(M) \leqq .4a, & 0 \leqq x_2(M) \leqq .1a, & M \in F_1. \end{array}$$

If $x = x_0 + x_1 - x_2$, then $x \in K_R$ and

$$|f(M) - x(M)| < .9a, \qquad \text{for all } M \in \mathfrak{M}.$$

This is a contradiction; hence $a = 0$ for any real-valued continuous $f(M)$.

If $f(M)$ is complex-valued, we can apply the result just proved to the real and imaginary parts of $f(M)$ separately.

References

1. I. Gelfand, *Normierte Ringe*, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 3–24.

2. I. Gelfand and G. Šilov, *Ueber verschiedene Methoden der Einfuehrung der Topologie in die Menge der maximalen Ideale eines normierten Ringes*, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 25–39.

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