A GENERALIZATION OF THE RUTT-ROBERTS THEOREM[1]

GAIL S. YOUNG, JR.

One of the more useful theorems of plane topology was proved virtually simultaneously by Rutt [**3**] and Roberts [**2**].[2] As modified by Moore [**1**, p. 296], it is essentially this: *Suppose that in the plane or the 2-sphere there exist two points, a and b; a collection, G, of continua whose union is a compact set, M, not containing a or b; and a continuum, C, such that the intersection of each two elements of G is precisely C. Then if no element of G separates a from b, neither does M.* This is, of course, a form of addition theorem. Even in 3-space this result is not true as stated, for the collection of circles given in rectangular coordinates by $x^2+y^2+z^2-x=0$, $az=bx$, for all a, b, satisfies all the conditions on G with respect to the points (1/2, 0, 0), (2, 0, 0), whereas their union separates these points. There is, however, a theorem concerning linking, which is valid in quite general spaces, and which reduces to the above theorem in the plane.

THEOREM 1. *Let S be a normal space acyclic in dimension $i+1$, and let Z^i be a cycle*[3] *on a compact subset K of S. Let the compact set M in $S-K$ be the union of a collection $G=\{C_\alpha\}$ of closed sets satisfying the following*: (1) *for every α, $Z^i\sim 0$ in $S-C_\alpha$*; (2) *there is a set C which is the intersection of each two elements of G and which links no $(i+1)$-cycle of S*; (3) *no closed set which is a union of elements of G links any $(i+1)$-cycle*; *and* (4) *any closed set which is the union of more than one element of G can be split into two closed proper subsets which are unions of elements of G. Then $Z^i\sim 0$ in $S-M$.*

The relation of this to the original result is perhaps clear except for condition (4). It is not difficult to show (cf. Moore [**1**, p. 296]) that if in the original theorem M separates a from b, then (4) holds. The conditions (2) and (3) follow from the fact that no continuum links a 1-cycle in the 2-sphere.

PROOF. Let M' be a closed subset of M which is the union of more than one element of G. Suppose that Z^i does not bound in $S-M'$. I show that then M' contains a proper closed subset M'', also the union of elements of G, such that Z^i does not bound in $S-M''$. By

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

[3] Čech cycles and homologies on compact sets and with field coefficients are used throughout.

hypothesis (4), M' is the union of two closed proper subsets, M_1 and M_2, both of which are unions of elements of G. The set $M_1 \cap M_2$ is either C or is a union of elements of G. Now suppose that neither M_1 nor M_2 links Z^i. Then for $k=1, 2$, there is a chain c_k^{i+1} in $S-M_k$, whose boundary is Z^i. Then $c_1^{i+1}-c_2^{i+1}$ is an $(i+1)$-cycle in $S-M_1 \cap M_2$, and by hypotheses (2) and (3) bounds there. But then from Wilder's generalization [5, p. 241] of the Alexander Addition Theorem, Z^i must bound in $S-M'$. Hence at least one of M_1 or M_2 links Z^i; let that one be M''.

Now it is quite clear that if each of a monotone collection of compact sets links Z^i, then so does their intersection; and that the intersection of a monotone collection of closed unions of elements of G is a union of elements of G, or is C. Hence if the theorem is false, by Zorn's lemma, there is a closed subset M^* of M which is irreducible with respect to the property of being a closed union of elements of G that links Z^i. (The possibility that $M^*=C$ can be immediately discarded by hypothesis (1).) Since M^* necessarily contains more than one element of G, it follows from the first paragraph of the proof that it is not a minimal closed union of elements of G linking Z^i, thus yielding a contradiction.

The hypotheses of this theorem are disappointingly complex if one hopes for a theorem as useful in higher dimensions as the original has been in the plane. However, I have examples to show that none of the hypotheses can be removed, or indeed much relaxed, and still leave the theorem true. In particular, for each $n>2$, I have an example in S^n of a compact set carrying a nonbounding $(n-1)$-cycle, and which is the union of a collection G of disjoint sets whose elements are points, arcs, and triods, with the property that for $0<i<n-1$ no closed union of elements of G carries a nonbounding i-cycle.

There is one case in which condition (4) can be replaced by rather natural conditions.

THEOREM 2. *Let the compact metric space S be the union of a collection G of closed sets such that there is a closed set C which is the intersection of each two elements of G. Suppose that G is upper semi-continuous in the sense that the union of all elements of G intersecting a compact subset of $S-C$ is closed.*[4] *Then S is the union of two closed proper subsets, each a union of elements of G.*

PROOF. If X is a compact set, not meeting C, and H is the collection of all sets which are the intersection of X with an element of G,

[4] In a compact space, if the elements of G are disjoint, this is equivalent to the ordinary definition of upper semi-continuity.

then H is upper semi-continuous in the ordinary sense. Hence H defines a continuous transformation $f: X \rightarrow Y$, where Y is the decomposition space of H (cf. Whyburn [**4**, pp. 125–127]). Given two proper closed subsets A_1, A_2 of Y, there are two proper closed subsets, B_1, B_2, of Y such that B_i contains A_i, and $B_1 \cup B_2 = Y$. The sets $f^{-1}(B_1)$, $f^{-1}(B_2)$ are proper closed subsets of X which are each unions of elements of H.

Now let M_n denote the set of all points of S at distance not less than $1/n$ from C, and let H_n denote the collection of intersections of M_n with elements of G. By the first paragraph, M_1 is the union of two closed proper subsets, N_{11} and N_{12}, each a union of elements of H_1. The union of all elements of H_2 that contain points of N_{1i}, $i=1, 2$, is closed, and is not all of M_2. Hence M_2 is the union of two closed proper subsets, N_{21} and N_{22}, with N_{2i} containing N_{1i}, and each a union of elements of H_2. We similarly define N_{31}, N_{32}, N_{41}, N_{42}, and so on. Now $C \cup \bigcup N_{j1}$ and $C \cup \bigcup N_{j2}$ are both closed proper subsets of S, and each is a union of elements of G.

This last result perhaps has most interest when C is empty and G is the collection of point-inverses for some continuous transformation. By placing various conditions on such a transformation, Theorem 1 yields a number of theorems, none of which, however, seem to settle any of the major outstanding problems of topology.

Bibliography

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University of Michigan