ON FINITENESS CONDITIONS FOR A CONVEX BODY

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1. Introduction. In certain problems of geometric number theory, such as the study of critical lattices, it is important to know when a finite convex body surrounding the origin will contain no lattice point other than the origin in its interior. In this brief note we shall discuss a situation in which it suffices to know that only a *finite* set of lattice points are not interior to the body. In the companion article [1][1] we shall show that the results of the present article enable us to reduce the determination of critical lattices in d dimensions to a finite number of steps.

2. Main theorem. *Let K be a convex body in d-dimensional euclidean space containing in its interior the hypersphere $\Sigma_d(r)$ of radius r and center at the origin* ($r<1$). Let us always consider the lattice of points with integral coordinates. *Then if K fails to contain in its interior those lattice points* (*other than the origin*) *in a second sphere $\Sigma_d(C_d/r^{d-1})$, then it will contain in its interior no lattice points at all except the origin. In fact K will then lie entirely within the second sphere.*

Proof. We shall establish the value $C_d = 2^{d-1}d\Gamma((d+1)/2)\pi^{-(d-1)/2}$.

We let (x_i) be an arbitrary point outside of $\Sigma_d(r)$, that is, we let $\Sigma x_i^2 = R^2 > r^2$, and we join this point to its negative by a line, namely $\Lambda(x_i)$, through the origin. We let $\Pi(x_i)$ be the $(d-1)$-dimensional hyperplane through the origin and perpendicular to this line, and we let $\Sigma_{d-1}(r; x_i)$ be the intersection of this plane with the solid sphere $\Sigma_d(r)$. For our immediate purposes we define $\kappa(r; x_i)$ as the (solid) cone formed by the segments joining each point of $\Sigma_{d-1}(r; x_i)$ with (x_i); and for later purposes we define $\mathrm{K}(r; x_i)$ as the (solid) cone with spherical base formed by joining with (x_i) each point of $\Sigma_d(r)$. Now $\mathrm{K}(r; x_i)$ will prove useful when we seek the best C_d, since this solid is a *minimal* body which any convex body K must contain in order to contain both $\Sigma_d(r)$ and (x_i). For our immediate purposes, we need only note that the convex body K will (easily) contain $\kappa(r; x_i)$ and that the latter body has the very handy value $\omega_{d-1}r^{d-1}R/d$ for its volume, where $\omega_k = \pi^{k/2}/\Gamma(1+k/2)$, the k-dimensional volume of the unit sphere in k dimensions.

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[1] Numbers in brackets are items or items and page numbers cited in the bibliography at the end.

We now note that $\kappa(r; x_i)$ will contain a lattice point (not the origin) if $R \geqq C_d/r^{d-1}$. This is a simple consequence of Minkowski's theorem on convex bodies [3, p. 60]. For the body formed by the union of $\kappa(r; x_i)$ and $\kappa(r; -x_i)$ is symmetric and has volume not less than 2^d. Hence there would have to be a lattice point (not the origin) in each of the two symmetric halves. From this the theorem follows easily.

3. **Best constant.** The preceding estimates can be improved slightly. For instance we may take into account the fact that the convex bodies, $\kappa(r; x_i)$ and its integral translates, can not pack space more densely than do the $(d-1)$-dimensional spheres (which are precisely their intersections with planes parallel to $\Pi(x_i)$). Then we immediately obtain an improvement of C_d to $C_d' = C_d\omega_{d-1}\gamma_{d-1}^{(d-1)/2}2^{-(d-1)}$ (where $\gamma_d^{1/2}$ is the maximum euclidean distance separating the closest points of any d-dimensional lattice of determinant unity). But using even the *lower* estimates [2, pp. 23–24] on γ_{d-1}, we find the improvement is not substantial, that is, as $d\rightarrow\infty$, $\log C_d \sim \log C_d' \sim (d/2) \log d$.

Going in the other direction we can show that there is a best constant for C_d as $r\rightarrow 0$. To recapitulate, our main theorem was deduced from *Minkowski's theorem*. It could also have been deduced from the *simultaneous approximation theorem*, as the remaining proof will suggest, but the constant C_d would not have been as good. The converse of Minkowski's theorem, or the *Minkowski-Hwlaka theorem* [4], is inapplicable since the problem is not affine invariant. The converse of the simultaneous approximation theorem, or (essentially) *Perron's transferal principle* [2, p. 67], is, however, applicable and we shall now look to it for a proof that a best C_d, which we shall call C_d^*, exists.

For the calculation of the best constant, we take a minimal convex body, which is "evasive" of lattice points, namely $K^* = K(r; \rho\theta_i/(\Sigma\theta_i^2)^{1/2})$, reaching to distance ρ from the origin in the direction $(\theta_1(=1), \theta_2, \cdots, \theta_d)$. The θ_k are chosen, from the Perron transferal principle, to span (say) a real algebraic field of degree d (for convenience, $0<\theta_k\leqq 1$). Then a constant g_d (called C_d in [2]) will exist with the property that "the forms $|p_1\theta_k - p_k|$, for $2\leqq k\leqq d$, admit a simultaneous approximation no better than $(g_d p_1^{1/(d-1)})^{-1}$," that is, for $p_1 > P_1(\epsilon)$, the inequality $|p_1\theta_k - p_k| \geqq ((g_d+\epsilon)p_1^{1/(d-1)})^{-1}$ will hold for some k depending on p_1 and ϵ, and satisfying $2\leqq k\leqq d$.

We now let ρ increase from r until it reaches the first value ρ_0 for which K^* has a lattice point (p_i) on its surface. If we took r small enough $(<r_0(\epsilon))$ then this lattice point would satisfy the inequality

$p_1 > P_1(\epsilon)$ by the irrationality of the θ_k when $k > 1$. If we let $\delta = [\Sigma_{i>j}(p_i\theta_j - p_j\theta_i)^2]^{1/2}/(\Sigma\theta_i^2)^{1/2}$ denote the distance from the lattice point (p_i) to the line $\Lambda(\theta_i)$, then easily $r > \delta$ (since r is the maximum distance from a point of K^* to $\Lambda(\theta_i)$), and $\delta > [p_1^{1/(d-1)}(g_d+\epsilon)(\Sigma\theta_i^2)^{1/2}]^{-1}$. Therefore using the trivial relationships $\rho_0 \geqq (\Sigma p_i^2)^{1/2} \geqq p_1$ and $\Sigma\theta_i^2 < d$, we find $\rho_0 > [(g_d+\epsilon)^{d-1}d^{(d-1)/2}r^{d-1}]^{-1}$. This gives us a distance that the convex body K^* can attain without containing an interior lattice point other than the origin. Thus it easily follows that $C_d' \geqq C_d^* \geqq c_d = (g_d d^{1/2})^{-(d-1)}$.

Even using very simple methods, we can improve c_d somewhat; but still, on the basis of present information on g_d [2, p. 72], we can show only that as $d \to \infty$, $\log C_d^*/(d \log d)$ lies between $1/2$ and -1. At any rate, the function C_d/r^{d-1}, appearing in the main theorem, is seen to contain the best possible power of r.

Bibliography

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