ON THE FINITE DETERMINATION OF CRITICAL LATTICES

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1. **Introduction.** We are concerned here with the determination of critical lattices with respect to a given finite convex body K containing the origin and symmetric to it. According to the usual terminology, a *critical lattice* is a K-admissible lattice of minimal determinant, while a *K-admissible lattice* is one all of whose points, except the origin, lie on or outside K. It is not hard to show [**5**, p. 13][2] that a critical lattice exists for every such K.

In d-dimensional space a lattice is determined by d^2 real variables or d vectors $\vec{\alpha_1}, \vec{\alpha_2}, \cdots, \vec{\alpha_d}$. If K is defined by a gauge function, in vector form $\Phi(x_i) \leqq 1$, then the problem is to minimize the determinant of the $\vec{\alpha_i}$ subject to the infinitude of side conditions

$$\Phi(p_1\vec{\alpha_1} + p_2\vec{\alpha_2} + \cdots + p_d\vec{\alpha_d}) \geqq 1 \tag{1}$$

where (p_i) runs through all integer sets except the origin. Now, as mentioned earlier it is possible to show the existence of a critical lattice of positive determinant, in fact, by the use of a compactness argument. We shall show, however, by means of a theorem in the companion paper [1], that the infinite set (1) can be reduced to a finite set whose number of inequalities depends only on d. Thus, for instance, we are assured that when K is bounded by algebraic surfaces, then among the (possibly continuous) set of critical lattices, a lattice with an algebraic basis can be found.

Results such as these are implied in the classical reduction theory of quadratic forms and of star bodies [2]. Indeed, Minkowski [**3**, pp. 51, 101] solved the problem for convex bodies explicitly when $d=2$ and 3, giving the most economical set of inequalities. The sets given here will be far from economical, but the methods will be comparitively simple and geometrically intuitive.

2. **Finite bases lemma.** *Let $\vec{\omega_1}, \vec{\omega_2}, \cdots, \vec{\omega_d}$ be a set of d independent vectors on the surface of the finite symmetric convex body K, and belonging to a K-admissible lattice. Then the lattice has a basis from the following finite set:*

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

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\begin{aligned}
&\vec{\Omega}_1 = \frac{b_{11}\vec{\omega}_1}{a_1},\\
(2)\qquad &\vec{\Omega}_2 = \frac{b_{21}\vec{\omega}_1}{a_1a_2} + \frac{b_{22}\vec{\omega}_2}{a_2},\\
&\cdots\cdots\cdots\cdots\cdots,\\
&\vec{\Omega}_d = \frac{b_{d1}\vec{\omega}_1}{a_1a_2\cdots a_d} + \frac{b_{d2}\vec{\omega}_2}{a_2\cdots a_d} + \cdots + \frac{b_{dd}\vec{\omega}_d}{a_d},
\end{aligned}
$$

where the a_i and b_{ij} are integers satisfying the conditions:

$$(2a)\qquad a_k > 0, \quad b_{kk} = 1, \quad |b_{kl}/a_k\cdots a_l| \leqq \frac{1}{2}a_l^{-1},$$

and

$$(2b)\qquad a_1a_2\cdots a_d \leqq d!.$$

This result is due to Minkowski [**4**, p. 189]; we therefore simply note some special aspects of the lemma which tie in with familiar results. In case there are more than d vectors (and their negatives) on the surface, let us choose $(\vec{\omega}_i)$ so as to produce a minimal positive determinant. Then for $d=2$ the basis can always be chosen as $(\vec{\omega}_1, \vec{\omega}_2)$ itself, while for $d=3$ it can be chosen as $(\vec{\omega}_i, \vec{\omega}_2, \vec{\omega}_3)$ or $(\vec{\omega}_1, \vec{\omega}_2, (\vec{\omega}_1+\vec{\omega}_2+\vec{\omega}_3)/2)$, as was known to Minkowski [**3**, p. 101]. If K is specialized to an ellipsoid, then for $d\leqq 3$ only the basis (ω_i) will occur.

3. **Reduction to finiteness.** In the search for critical lattices, we first restrict the K-admissible lattices to the subclass having d independent vectors on the surface of K. This is no restriction since every K-admissible lattice can be contracted to a K-admissible lattice with the further property. We then let $(\vec{\omega}_i)$ be such a set of d independent vectors. By the finite bases lemma, the lattice has as its basis a set $(\vec{\Omega}_i)$ chosen from among the finite collection in (2).

Then the conditions (1), subsidiary to the minimizing of the determinant of the $\vec{\Omega}_i$, can be rewritten as $\Phi(\sum p_i\ \vec{\Omega})\geqq 1$, where (p_i) are still all integral lattice points, not the origin, and where equality is achieved when the argument of Φ is one of the vectors $\vec{\omega} = \sum_{j=1}^{i} a_{ij}\vec{\Omega}_j$, from (2). Now the convex body K determines a *conjugate* convex body K_0 in the p-space by means of the inequality in the p_i: $\Phi(\sum p_i\ \vec{\Omega}_i)\leqq 1$. Furthermore K_0 contains the 2^d-hedron determined by the vectors $\pm(a_{11}, 0, 0, \cdots, 0)$, $\pm(a_{21}, a_{22}, 0, \cdots, 0)$, $\cdots$,

$\pm(a_{d1}, a_{d2}, a_{d3}, \cdots, a_{dd})$. But these 2^d-hedra are finite in number, one for each basis, and they contain the origin and hence a hypersphere of radius ρ_d, depending on d alone. To be more specific, using the estimates $a_{11} \cdots a_{dd} < d!$ and $a_{jj} \geqq |a_{kj}|$ for $k > j$ (an easy consequence of (2a)), we find that for instance $\rho_d \geqq d^{-1/2}(d!)^{-3/2}$. Hence by the main theorem of the companion paper [1], k will contain no p-integral lattice points, other than the origin, in its interior if it fails to contain those in a hypersphere of radius $C_d \rho_d^{-(d-1)}$.

Thus the system of inequalities (1) can now be replaced by the finite collection of the finite systems (one system for each basis)

$$\text{(1a)} \quad \begin{aligned} &\Phi\left(\sum_{i=1}^{k} a_{ik}\vec{\Omega_k}\right) = 1, && 1 \leqq k \leqq d, \\ &\Phi\left(\sum_{i=1}^{1} p_i\vec{\Omega_i}\right) \geqq 1, \text{ for } 0 < \left(\sum p_i^2\right)^{1/2} < C_d \rho_d^{-(d-1)} && (p_i \text{ integral}). \end{aligned}$$

In principle, of course, we select the basis whose system gives the smallest determinant; and in each system we would have had to change some inequalities to equality and to ignore many more. When $d > 3$, however, there seems to be no easy way to determine a minimal system.

Bibliography

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