TWO MAPPING PROPERTIES OF SCHLICHT FUNCTIONS

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The mapping properties we shall prove hold for the normalized exterior mapping function of a simple analytic curve. Let C be a simple analytic curve in the z-plane and designate its exterior by D. The normalized exterior mapping function of C is the analytic function $w=f(z)$ which is uniquely determined by the conditions that (i) it is regular in D except for a simple pole at $z=\infty$, (ii) its power series expansion about $z=\infty$ has the normalization

$$(1) \qquad w = z + a_0 + \frac{a_1}{z} + \cdots ,$$

and (iii) it maps D in a 1–1 manner onto the exterior of a circle Σ, $|w| = \rho$.

THEOREM I. *Let C be a simple analytic curve, and designate its exterior by D. Let $f(z)$ be the normalized exterior mapping function of C. Let σ be a circle with center z_0, whose closed interior lies in D. Then $F(z) = f(z)/(z-z_0)$ maps σ onto a curve in the w-plane that is star-shaped from the point $w=0$.*

PROOF. A curve Γ is star-shaped with respect to a point A in its interior if it is a simple curve, and if each point of Γ can be connected to A by a straight line lying in the interior of Γ. Let σ have radius r, and let Z be a point on σ. Then $Z-z_0=re^{i\theta}$. Let $F(Z)=Re^{i\phi}$. For the image of σ to be star-shaped, $d\phi/d\theta$ must not vanish, and be of constant sign for $0 \leqq \theta < 2\pi$. Since $F(z)$ has a simple pole in σ, and otherwise is regular and nonzero there, ϕ decreases by 2π when θ increases by 2π, so $d\phi/d\theta$ must be negative for some value θ', $0 \leqq \theta' < 2\pi$. We now show that it is negative for each value of θ in the interval.

We first express $d\phi/d\theta$ at a point Z on σ in terms of $f(Z)$. Start with

$$(2) \qquad \begin{aligned} \frac{d\phi}{d\theta} &= \frac{d}{d\theta} \operatorname{Im} \log F(Z) \\ &= \frac{d}{d\theta} \operatorname{Im} ((\log f(Z) - \log (Z - z_0)). \end{aligned}$$

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Differentiate, to obtain

$$\frac{d\phi}{d\theta} = \operatorname{Im}\left(\frac{f'(Z)}{f(Z)}\frac{dZ}{d\theta} - \frac{1}{Z - z_0}\frac{dZ}{d\theta}\right). \tag{3}$$

Substituting $dZ/d\theta = i(Z-z_0)$, we obtain

$$\begin{aligned} \frac{d\phi}{d\theta} &= \operatorname{Im}\left(i(Z - z_0)\frac{f'(Z)}{f(Z)} - i\right) \\ &= \operatorname{Re}\left((Z - z_0)\frac{f'(Z)}{f(Z)} - 1\right). \end{aligned} \tag{4}$$

We now use the Cauchy integral formula to obtain a representation for $f'(Z)/f(Z)$. Since $f'(z)/f(z)$ is regular in D, and tends to zero as $z \to \infty$, and since each point Z lies in D, for a fixed Z we have

$$\frac{f'(Z)}{f(Z)} = \frac{1}{2\pi i}\int_{C^-} \frac{1}{z - Z}\frac{f'(z)}{f(z)}\,dz. \tag{5}$$

Let $f(z) = \rho e^{i\alpha}$ when z is on C, and indicate the inverse of $w = f(z)$ by $z = z(w)$. Then $(f'(z)/if(z))\,dz = d\alpha$ and from (5) we have

$$\frac{f'(Z)}{f(Z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{Z - z(\rho e^{i\alpha})}\,d\alpha. \tag{6}$$

Substituting (6) in (4), we obtain

$$\frac{d\phi}{d\theta} = \operatorname{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{Z - z_0}{Z - z(\rho e^{i\alpha})}\,d\alpha - 1\right). \tag{7}$$

Since $(1/2\pi)\int_0^{2\pi} d\alpha = 1$, this can be written

$$\begin{aligned} \frac{d\phi}{d\theta} &= \operatorname{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{Z - z_0}{Z - z(\rho e^{i\alpha})} - 1\right)d\alpha\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{z(\rho e^{i\alpha}) - z_0}{Z - z(\rho e^{i\alpha})}\right)d\alpha. \end{aligned} \tag{8}$$

The integrand in (8) is a continuous function of α since the circumference of σ is bounded from C. Hence, to prove $d\phi/d\theta < 0$, it suffices to show that the integrand in (8) is negative for α, $0 \leqq \alpha < 2\pi$. Indeed, let α_1 be a value in this interval, and let $z(\rho^{i\alpha_1}) = z_1$. Then

$$\operatorname{Re}\frac{z_1 - z_0}{Z - z_1} < 0 \tag{9}$$

if

$$\operatorname{Re}\frac{Z - z_1}{z_1 - z_0} < 0.$$

We write

$$\operatorname{Re}\frac{Z - z_1}{z_1 - z_0} = \operatorname{Re}\frac{Z - z_0 + z_0 - z_1}{z_1 - z_0} = \operatorname{Re}\frac{Z - z_0}{z_1 - z_0} - 1. \tag{11}$$

Since $|Z - z_0| < |z_1 - z_0|$, we have

$$\operatorname{Re}\frac{Z - z_0}{z_1 - z_0} - 1 \leqq \left|\frac{Z - z_0}{z_1 - z_0}\right| - 1 < 0. \tag{12}$$

From (12) it follows that the integrand in (8) is negative for each value of Z on σ, and the theorem is proved.

THEOREM II. *Let C be a simple analytic curve, and designate its exterior by D. Let $f(z)$ be the normalized exterior mapping function of C. Let Z_1 and Z_2 be two points in D. If Z_1 and each point of C lie on the same side of the perpendicular bisector, L, of the line joining Z_1 and Z_2, then $|f(Z_1)| < |f(Z_2)|$.*

PROOF. Representation (6) for $f'(Z)/f(Z)$ is valid for Z in D, hence by integration

$$\log f(Z_i) = \frac{1}{2\pi}\int_0^{2\pi} \log (Z_i - z(\rho e^{i\alpha}))d\alpha \qquad (i = 1, 2). \tag{13}$$

It then follows that

$$\log\left|\frac{f(Z_1)}{f(Z_2)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{Z_1 - z(\rho e^{i\alpha})}{Z_2 - z(\rho e^{i\alpha})}\right| d\alpha. \tag{14}$$

The integrand in (14) is a continuous function of α. From the hypothesis it follows that $|(Z_1 - z(\rho e^{i\alpha}))/(Z_2 - z(\rho e^{i\alpha}))| \leqq 1$. The inequality must hold for some α, for otherwise $z(\rho e^{i\alpha})$ would be restricted to a line and C would not be simple. Hence the integral in (14) is negative and $|f(Z_1)| < |f(Z_2)|$. This completes the proof.

Theorem I provides a complement to results stated by Pólya-Szegö [1, pp. 104–105].[1] It also gives part of the domain of schlichtness of the ratio of two schlicht functions.

At the suggestion of the referee we shall discuss Theorem II*, which is Theorem II under the more general hypothesis that D is an arbitrary, simply-connected domain containing $z = \infty$, its bound-

[1] This refers to the bibliography at the end of the paper.

ary set C is not an analytic curve (since this case is covered by Theorem II), and $f(z)$ is the analytic function with normalization (1) which maps D in a 1–1 manner onto the exterior of a circle Σ, $|w| = \rho$. We lose no generality in taking L to be the real axis, in which case Z_1 and Z_2 can be replaced by Z and Z^*,[2] where Im $Z>0$. If C is in the half-plane Im $z>0$, a level curve C_1 of $w=f(z)$ also lies in this half-plane with Z in its exterior, and this curve can be used in place of C in the proof used for Theorem II. This shows that $|f(Z)| < |f(Z^*)|$. If C lies in the half-plane Im $z \geqq 0$, touching L, the level curves of $f(z)$ will all be cut by L. Select a sequence, $\{C_k\}$, whose exteriors exhaust D. In the proof of Theorem II, replace C by C_n to obtain

$$\log\left|\frac{f(Z)}{f(Z^*)}\right| < \epsilon_n, \tag{15}$$

where $\epsilon_n > 0$ and ϵ_n tends to zero as n tends to ∞! Hence

$$|f(Z)| \leqq |f(Z^*)|. \tag{16}$$

We now show that equality in (16) implies that C coincides with L. Suppose $|f(Z_1)| = |f(Z_1^*)|$, where Z_1 is an interior point of D, satisfying Im $Z_1 > 0$. Since (16) holds in a neighborhood of Z_1, $f(z)$ must satisfy the functional relationship

$$f(z) = e^{i\alpha}(f(z^*))^*, \tag{17}$$

where α is a constant, $0 \leqq \alpha < 2\pi$. Letting $x \to \infty$, $z = x + iy$, we see that, because of normalization (1), α must equal 0. But then, interpreted geometrically, (17), with $\alpha = 0$, implies that C is symmetric about L. Since C lies in the half-plane Im $z \geqq 0$, it must coincide with L. Since C is the boundary of a simply-connected domain, it is an interval of L. Hence

$$f(z) = \frac{(az+b) + ((az+b)^2 - 1)^{1/2}}{2a} \tag{18}$$

where a, b are real, $a > 0$, and the branch is determined by choosing the positive sign of the radical for z positive and sufficiently large.

Bibliography

1. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, New York, Dover, 1945.

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[2] Z^* denotes the complex conjugate of Z.