FRÉCHET AND KERÉKJÁRTÓ EQUIVALENCE

EARL J. MICKLE

1. **Introduction.** Let P_1, P_2 be two homeomorphic Peano spaces, let $\mathfrak{P}$ be a metric space with metric $d(p', p'')$, and let T_1, T_2 be continuous mappings from P_1, P_2 respectively into $\mathfrak{P}$. The reader is referred to Radó [1][1] for definitions and concepts used here.

T_1 and T_2 are called Fréchet equivalent, written for brevity $T_1 \sim T_2(F)$, if for every $\epsilon > 0$ there is a homeomorphism h_ϵ from P_1 onto P_2 such that the inequality $d[T_1(p_1), T_2 h_\epsilon(p_1)] < \epsilon$ holds for each point $p_1 \in P_1$.

T_1 and T_2 are called Kerékjártó equivalent, written for brevity $T_1 \sim T_2(K)$, if T_1 and T_2 admit of simultaneous monotone-light factorizations with the same middle-space and the same light factor.

We have (see Radó [1, p. 60]) that $T_1 \sim T_2(F)$ implies that $T_1 \sim T_2(K)$ whereas simple examples (see Radó [1, p. 60]) show that the converse is generally false. However, in certain special cases Kerékjártó equivalence does imply Fréchet equivalence. Two very important special cases used in surface area theory are the following.

(a) If P_1, P_2 are 2-cells and the common middle-space of the simultaneous monotone-light factorizations is a 2-cell, then (see Radó [1, p. 74]) $T_1 \sim T_2(K)$ implies that $T_1 \sim T_2(F)$.

(b) If P_1, P_2 are 2-spheres and the common middle-space of the simultaneous monotone-light factorizations is a 2-sphere, then (see Youngs [2]) $T_1 \sim T_2(K)$ implies that $T_1 \sim T_2(F)$.

If P is a Peano space, then the uniform limit of a sequence of homeomorphisms from P onto P is a continuous monotone mapping from P onto P. The results given in (a) and (b) above depend on the converse of this statement. That is to say, if P is a 2-cell or a 2-sphere and m is a continuous monotone mapping from P onto P, then m is the uniform limit of a sequence of homeomorphisms from P onto P (see Radó [1, pp. 71–72]). For extension of these results see Youngs [3]. In this note we give an example of a continuous monotone mapping of a 3-cell onto itself which is not the uniform limit of a sequence of homeomorphisms. This particular example enables us to show that in the 3-cell case Kerékjártó equivalence never implies Fréchet equivalence.

2. **Preliminary lemmas.** Let P be a Peano space, let I be the closed

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

unit interval $0 \leqq t \leqq 1$, let I^0 be the open interval $0 < t < 1$, let T be a continuous mapping from P onto I, and let (boldface arrows indicate a mapping onto) $T = lm$, $m: P \twoheadrightarrow \mathfrak{M}$, $l: \mathfrak{M} \twoheadrightarrow I$, be a monotone-light factorization of T. For $A = T^{-1}(0)$, $B = T^{-1}(1)$, $A^* = l^{-1}(0)$, $B^* = l^{-1}(1)$ we set α equal to the number of components O of $P - (A + B)$ for which $T(O) = I^0$ and α^* equal to the number of components O^* of $\mathfrak{M} - (A^* + B^*)$ for which $l(O^*) = I^0$.

LEMMA 1. *Under the above conditions* $0 < \alpha = \alpha^* < \infty$.

PROOF. We have a simple arc γ with end points in A and in B and for which $\gamma^0(A + B) = 0$, where γ^0 is γ minus its end points. Then γ^0 is in some component O of $P - (A + B)$ and $T(O) \supset T(\gamma^0) = I^0$. Therefore $\alpha > 0$.

Assume that there is an infinite number $O_1, O_2, \cdots$ of components of $P - (A + B)$ such that $T(O_n) = I^0$. Then the closures C_n of O_n are continua and $T(C_n) = I$. We have a subsequence C_{n_j} which converges to a continuum C for which $T(C) = I$. Let O be a component of $P - (A + B)$. All but at most one of the C_{n_j} are in $P - O$. Hence $C \subset A + B$ and $I = T(C) \subset T(A + B) = I - I^0$. This is a contradiction and hence $0 < \alpha < \infty$.

By applying the above result to the mapping l we have $0 < \alpha^* < \infty$. Since $T = lm$ and m is monotone, it follows that $\alpha = \alpha^*$.

Now let P_1, P_2 be homeomorphic Peano space, let T_1, T_2 be continuous mappings from P_1, P_2 respectively onto I, and let α_1 and α_2 be defined as above.

LEMMA 2. *Under the above conditions if* $T_1 \sim T_2(F)$, *then* $0 < \alpha_1 = \alpha_2 < \infty$.

PROOF. Since $T_1 \sim T_2(F)$, we have simultaneous monotone-light factorizations of the form $T_1 = lm_1$, $m_1: P_1 \twoheadrightarrow \mathfrak{M}$, $l: \mathfrak{M} \twoheadrightarrow I$, $T_2 = lm_2$, $m_2: P_2 \twoheadrightarrow \mathfrak{M}$, $l: \mathfrak{M} \twoheadrightarrow I$. Let α^* be defined as above for the mapping l. By Lemma 1, $\alpha_1 = \alpha^*$ and $\alpha_2 = \alpha^*$. Therefore $0 < \alpha_1 = \alpha_2 < \infty$.

3. A particular continuous monotone mapping. Let V: $0 \leqq x^2 + y^2 + z^2 \leqq 1$ be the solid unit sphere with center at the origin in xyz-space. Then V can be given in spherical coordinates

$$V:\ x = r \cos\theta \sin\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r\cos\phi,$$
$$0 \leqq r \leqq 1,\ 0 \leqq \theta \leqq 2\pi,\ 0 \leqq \phi \leqq \pi,$$

where, for a point (x, y, z), r is the distance of the point from the origin, θ is the angle between the positive x-axis and the line from the origin to the point $(x, y, 0)$, and ϕ is the angle between the positive

z-axis and the line from the origin to the point.

THEOREM 1. *There exists a continuous monotone mapping $m: V \rightarrow V$ such that m is not the uniform limit of a sequence of homeomorphisms from V onto V.*

PROOF. Using spherical coordinates let

$$V': x = r' \cos \theta' \sin \phi', \qquad y = r' \sin \theta' \sin \phi', \qquad z = r' \cos \phi',$$
$$0 \leqq r' \leqq 1,\ 0 \leqq \theta' \leqq 2\pi,\ 0 \leqq \phi' \leqq \pi/2,$$

be the solid hemisphere $x'^2+y'^2+z'^2 \leqq 1$, $z' \geqq 0$, in $x'y'z'$-space. We map V' continuously onto V by the relations $r=r'$, $\theta=\theta'$, $\phi=2\phi'$. Call this mapping M. For $(x, y, z) \neq (0, 0, -1 \leqq z < 0)$, $M^{-1}(x, y, z)$ is a single point. For $(x, y, z) = (0, 0, -1 \leqq z < 0)$, $M^{-1}(0, 0, z)$ is the circle $x'^2+y'^2=z^2$ in the $x'y'$-plane. Thus M is a continuous monotone mapping from V' onto V. Let h be any homeomorphism from V' onto V. Then $h(0, 0, 0)$ is a point on the surface of V and since $M(0, 0, 0)$ is the point $(0, 0, 0)$, it follows that the distance between $h(0, 0, 0)$ and $M(0, 0, 0)$ is one. Therefore M is not the uniform limit of a sequence of homeomorphisms from V' onto V. If h^* is any homeomorphism from V onto V', then the same statement applies to the monotone mapping $m = Mh^*$ from V onto V.

4. **Main result.** For the solid sphere V given in §3 we shall denote by S the surface $x^2+y^2+z^2=1$ and for the solid hemisphere V' given in §3 we shall denote by H the surface $x'^2+y'^2+z'^2=1$, $z' \geqq 0$, by K the surface $0 \leqq x'^2+y'^2 \leqq 1$, $z'=0$ and by L the surface $H+K$.

THEOREM 2. *Let T be a continuous mapping from V into a metric space $\mathfrak{P}$. If $T(V)$ is not a single point, then there is a continuous mapping T^* from V into $\mathfrak{P}$ such that $T \sim T^*(K)$ but such that T and T^* are not Fréchet equivalent.*

PROOF. Let $T(0, 0, 0) = p_0$ and $T(0, 0, -1) = p_1$. We may assume without loss of generality that $p_0 \neq p_1$. Let M be the continuous monotone mapping from V' onto V defined in Theorem 1. Set $T' = TM$. Then $T \sim T'(K)$. We assert that T and T' are not Fréchet equivalent. Assume that $T \sim T'(F)$. Let T_S denote the mapping T operating from the surface S, and let T'_H, T'_K, and T'_L denote the mapping T' operating respectively from H, K, and L. Since any homeomorphism from V' onto V takes L onto S, the relation $T \sim T'(F)$ implies that $T_S \sim T'_L(F)$.

Case 1. $p_0 \notin T(S)$. Then $T'(0, 0, 0) = T(0, 0, 0) = p_0 \notin T(S) = T_S(S)$. Thus $T(S)$ and $T'(L)$ are not the same point set and T_S and T'_L are not Fréchet equivalent. Therefore T and T' are not Fréchet equivalent.

Case 2. $p_0 \in T(S)$, $\mathcal{P}$ *is the unit interval* $I: 0 \leqq t \leqq 1$ *with* $p_0 = 0$ *and* $p_1 = 1$. Let α_S, α_L, α_H, and α_K be defined as in §3 respectively for the mappings T_S, T'_L, T'_H, and T'_K. If M_H denotes the mapping M operating from H, then $T'_H = T_S M_H$ and, since M_H is monotone, it follows that $\alpha_S = \alpha_H$. Since $T'_L(x', y', 0) = 1$ for $x'^2 + y'^2 = 1$, it follows that $\alpha_L = \alpha_H + \alpha_K$. Since $\alpha_K > 0$, we have $\alpha_S < \alpha_L$. Therefore $\alpha_S \neq \alpha_L$ and, by Lemma 2, T_S and T'_L are not Fréchet equivalent. Therefore T and T' are not Fréchet equivalent.

Case 3. $p_0 \in T(S)$, $\mathcal{P}$ *an arbitrary metric space.* For the metric $d(p', p'')$ let $t = f(p) = d(p, p_0)/[d(p, p_0) + d(p, p_1)]$. Then f is a continuous mapping from $\mathcal{P}$ onto I and $T \sim T'(F)$ implies that $fT \sim fT'(F)$. Since $fT(0, 0, 0) = f(p_0) = 0$ and $fT(0, 0, -1) = f(p_1) = 1$, fT and fT' satisfy the conditions of Case 2. Thus fT and fT' are not Fréchet equivalent. Therefore T and T' are not Fréchet equivalent.

Now let h be any homeomorphism from V onto V' and set $T^* = T'h$. Then $T' \sim T^*(F)$ and hence T and T^* are not Fréchet equivalent.

Bibliography

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The Ohio State University