THE CENTROID OF A CONVEX BODY[1]

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1. **Introduction.** Let C be a closed planar convex body and let x be an interior point of C. Let the ratio into which x divides a chord of C passing through it be the ratio of the larger (or equal) segment to the whole chord. There is a maximum such ratio for each point x. Let r^* be the minimum of these maximum ratios for all points x in C. We call r^* the *critical ratio* and any point x^* which divides a chord in the critical ratio a *critical point*. B. H. Neumann[2] has proved for any planar convex body that there is a unique critical point and that the critical ratio satisfies the inequality $1/2 \leqq r^* \leqq 2/3$. The value $r^* = 1/2$ is achieved only for bodies with central symmetry and $r^* = 2/3$ is assumed only for triangles in which case the centroid of the triangle is the critical point. There are at least three chords which the critical point divides in the ratio r^*.

We here give proof that $1/2 \leqq r^* \leqq 2/3$ for closed planar convex bodies which generalizes readily to $1/2 \leqq r^* \leqq n/(n+1)$ for n-dimensional closed convex bodies.

2. **The planar case.** For a point x contained in the interior of a planar closed convex body we define the ratio r into which it divides a chord as the ratio of the larger segment to the whole chord. Hence $1/2 \leqq r < 1$. We prove now the following theorem.

THEOREM I. *If the centroid of a closed convex planar body is a point of trisection of any chord, then the body is a triangle. Furthermore, the centroid of every planar convex body divides every chord through it in a ratio less than or equal to $2/3$.*

PROOF. Let the convex body C as required be given. Let us indicate in order the construction of Fig. 1. The centroid x and the chord it trisects with end points b_1 and b_2 are given so that $b_1x/b_1b_2 = 1/3$. Choose now a line m_1 through b_1 which separates the plane into two halves, the open part of one of the halves not containing any points of C, the other closed half containing all points of C. This is always possible for convex bodies. Next, parallel to m_1 through x draw the line m_2. As the centroid x is an interior point of C there are two boundary points b_3 and b_4 of C on m_2 such that b_3b_4 contains x. Next,

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[2] *On some affine invariants of closed convex regions*, J. London Math. Soc. vol. 14, pp. 262–272.

construct a triangle by extending b_2b_3 and b_2b_4 until they meet m_1 in points a_1 and a_2 respectively. Consider now the regions marked I and II. No points of C can be in these regions for if so, b_3 or b_4 would be interior to C—that is, in a triangle containing b_2 and b_1 as two vertices and the other vertex in the region I or II. Hence, all points of C lie either above m_2 (as shown in the figure) or in the trapezoid $a_1a_2b_4b_3$. Now, consider the centroid y of the triangle $a_1a_2b_2$. It lies on

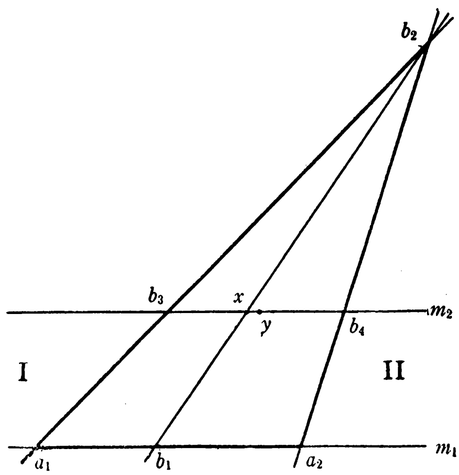

FIG. 1

the line m_2. We know that C contains all the points of the triangle $b_2b_3b_4$ by the convexity condition. The moment about m_2 of $b_2b_3b_4$ is equal to the negative of the moment about m_2 of the trapezoid $a_1a_2b_4b_3$ as m_2 contains the centroid of the triangle $a_1a_2b_2$. Hence, as x is the centroid of C, C must contain the entire area of the trapezoid to give the zero moment about m_2. But, C cannot contain any more than the trapezoid below m_2 and hence it cannot have a greater area than that of $b_3b_4b_2$ above m_2. Hence C is exhibited as the triangle $a_1a_2b_2$.

Now a slight adjustment of the above proof shows that if a supposed centroid x of a closed planar convex body C divides a chord in a ratio greater than $2/3$ then the same construction with m_2 through x (but not through y) shows that C cannot have a large enough trapezoidal area below m_2 to "balance" the triangle $b_3b_4b_2$ above m_2 which is in C. Hence, the centroid of a convex body in the plane divides no chord in a ratio greater than $2/3$. This concludes the proof of the theorem.

COROLLARY TO THEOREM I (NEUMANN). *The critical ratio r^* of a closed planar convex body satisfies the inequality* $1/2 \leqq r^* \leqq 2/3$.

PROOF. We have shown that the centroid of a closed planar convex body has a maximum ratio r no larger than $2/3$ and that this value is obtained in a triangle. Hence, $r^* \leqq 2/3$ for any planar convex body. By definition $r^* \geqq 1/2$.

3. n-dimensional case. We define the critical ratio r^* as before and we state the theorem (for $n \geqq 2$).

THEOREM II. *A closed n-dimensional convex body, the centroid of which divides a chord in the ratio* $r = n/(n+1)$, *is an n-dimensional hypercone based on a closed* $(n-1)$*-dimensional convex body. Furthermore, the critical ratio* r^* *ranges over the closed interval* $[1/2, n/(n+1)]$ *for all possible n-dimensional convex bodies.*

PROOF. The proof follows the exact analogue of the proof of Theorem I. Instead of lines m_1 and m_2 we have parallel hyperplanes and in place of points b_3, b_4, a_1, and a_4 we have $(n-2)$-dimensional boundaries of $(n-1)$-dimensional closed convex bodies and the argument otherwise proceeds formally as in the proof of Theorem I.

4. Concluding remarks. We note that in three dimensions if the centroid of a convex body divides two distinct chords in the ratio $3/4$ that the body is a tetrahedron. One would expect that a generalization of Neumann's theorem would state that there is a unique critical point x^* associated with every 3-dimensional convex body and that this point divides at least four chords in the critical ratio r^*. If so, one might expect similar results for higher dimensions. We have not proved such generalizations.[3]

One might take the view, from Neumann's results, that the centroid of a triangle trisects three chords because it is the critical point—not because it is the centroid. At one time we conjectured that the centroid of a planar convex body might be its critical point. However, this is disproved by a semicircle. The centroid of a semicircle of "radius" unity is $4/3\pi$ from the "center" while the critical point is also on the chord of symmetry at a distance $2^{1/2}-1$ from the center. The critical ratio in this case is $2-2^{1/2}$.

One is led to ask then the following questions: 1. What is the upper bound for the difference $r_c - r^*$ where r_c is the maximum ratio into

[3] Professor Andrew Sobczyk has given an example of a three-dimensional body in which there is more than one critical point. This will be one of the topics covered in a paper now being written by Professor Sobczyk and the author.

which the centroid divides chords and does a convex body exist in which this bound is achieved? 2. What other properties characterize the class of convex bodies in which $r_c = r^*$?

Since writing this note the author has discovered that a generalizable proof of Neumann's inequality of a different character was given by Wilhelm Suess in a note entitled *Ueber eine Affininvariante von Eibereichen*, published in Archiv fuer Mathematik vol. 1 (1948) pp. 127–128. One error in statement appears in this note—namely, that if the critical ratio is $n/(n+1)$ that the body is a simplex. As we have noted, the body is a convex hypercone but not necessarily a simplex. It may be true, although we have not proved it, that if the centroid of a convex body in E_n divides $n-1$ chords in the ratio $n/(n+1)$ that the body is a simplex.

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