ON THE EMBEDDING OF TOPOLOGICAL SEMIGROUPS AND INTEGRAL DOMAINS

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1. Introduction. In the following the topological aspects of the classical theory of embedding semigroups and integral domains are considered. §2 is a summary of the relevant topological and algebraic concepts and theorems which are employed in the remainder of the paper. In §3, the general theory is developed and applied to certain special cases. §4 is devoted to the proof that a commutative semigroup with a cancellation law is a group if it is compact. In §5 there is a discussion of metric semigroups and in §6 a more extended treatment is given for complete separable metric semigroups.

2. Algebraic and topological preliminaries. A commutative semigroup with identity e and cancellation law will be denoted by S. In the Cartesian product $S \times S$ let R be the relation defined by the statement: $(a, b)R(c, d)$ if and only if $ad = bc$. The algebraic quotient structure $S \times S/R$ will be denoted by $Q(S)$. The canonical mapping of $S \times S$ onto $Q(S)$ will be denoted by ϕ, and the set $\phi(S \times e)$ by $S^{\#}$. We shall designate integral domains by I and their multiplicative semigroups by I^*; R, $I \times I^*/R = Q(I)$, ϕ, and $I^{\#}$ will have analogous meanings.

A topological semigroup is a semigroup with a Hausdorff topology with respect to which multiplication is continuous in both variables. Topological groups and fields are the conventional entities. A topological integral domain is one whose additive group is topological and in which multiplication is continuous in both variables. In the case of a topological semigroup or integral domain, the relation R will be called *open* in case, for any open set U of $S \times S$ (of $I \times I^*$), the *saturation* of U, $\phi^{-1}(\phi(U))$, is open. We shall call S (I) *embeddable* or *weakly embeddable* in $Q(S)$ ($Q(I)$) if (a) $Q(S)$ ($Q(I)$) is a topological group (field) and (b) $S^{\#}$ ($I^{\#}$) in the induced topology is a topological or continuous image of S (I).

A topology on a given set will be called *weaker* than a second topology on the same set if every set that is open in the first topology is also open in the second. The second topology is then said to be *stronger* than the first. Thus weak embeddability of S or I means that the topology induced on $S^{\#}$ or $I^{\#}$ is weaker than the original.

We observe that if A is a topological space and if ϕ is a mapping of A onto B, then there exists a unique strongest topology on B with respect to which ϕ is continuous.

THEOREM 1. *R-equivalence classes in $S \times S$ $(I \times I^*)$ are closed sets in the product topology.*

PROOF. Let A be an equivalence class and assume (a, b) is in $\overline{A}$ (the closure of A in $S \times S$). Let U_a, U_b be any neighborhoods of a, b in S. Then (U_a, U_b) is a neighborhood of (a, b) in $S \times S$ and consequently (U_a, U_b) meets A. Fix (c, d) in A. If it is false that $(a, b)R(c, d)$, then $ad \neq bc$, and hence there are neighborhoods U_a and U_b such that dU_a and cU_b do not meet. But then there is an element (u_a, u_b) in A and in (U_a, U_b), and thus $u_a d = u_b c$. This is a contradiction. The proof for integral domains is identical.

We state the following theorem without proof.

THEOREM 2. *If the relation R is open and if $Q(S)$ $(Q(I))$ is endowed with the strongest topology with respect to which ϕ is continuous, then the mapping ϕ is open, and conversely.*

3. **Topology of quotient structures.** We shall now consider a criterion for the openness of the relation R. Since for any collection of sets in $I \times I^*$ the saturation of their union is the union of their saturations, the openness of R is clearly equivalent to the openness of the saturations of sets of the form (U_a, U_b). In fact, it is easily seen that openness is equivalent to the following: corresponding to any neighborhood of zero U and equivalent pairs of points (a, b) and (c, d) in $I \times I^*$, there exists a neighborhood of zero V such that every point of $(c + V, d + V)$ is equivalent to some point of $(a + U, b + U)$. A further simplification is possible. Because of the continuity of multiplication, if $a \in I$ and if W is any open set in I, then the set $[x : ax \in W]$ is open. It follows trivially that if x is a nonzero element of I and if (c, d) is any element of $I \times I^*$, then if $a = cx$ and $b = dx$, the conditions of the preceding criterion are satisfied, with $V = [y : xy \in U]$. Since any two equivalent pairs, (a, b) and (c, d), are equivalent to a "common multiple," $(ad, bd) = (bc, bd)$, it is possible to formulate the following criterion.

THEOREM 3. *The relation R is open if and only if, corresponding to a, b, and c in I, where $b \neq 0$ and $c \neq 0$, and a neighborhood of zero U, there exists a neighborhood of zero V such that every point of $(ac + V, bc + V)$ is equivalent to some point of $(a + U, b + U)$.*

For a commutative semigroup with cancellation law this criterion

takes the form: *The relation R is open if and only if, corresponding to elements a, b, and c and neighborhoods U_a and U_b, there exist neighborhoods U_{ac} and U_{bc} such that every point of (U_{ac}, U_{bc}) is equivalent to some point of (U_a, U_b).*

In case I is a metric space (or one whose topology can be described by the convergence of simple sequences) the criterion for openness of R can be stated as follows: *The relation R is open if and only if for arbitrary null sequences a_n, b_n and arbitrary a, $b(\neq 0)$, $c(\neq 0)$ one can find null sequences d_n, e_n with the property $(a+d_n, b+e_n)R(ca+a_n, cb+b_n)$.* A similar observation applies to S.

THEOREM 4. *If R is open and if $Q(S)$ $(Q(I))$ is endowed with the strongest topology with respect to which ϕ is continuous, then $Q(S)$ $(Q(I))$ is a topological group (field).*

PROOF. We prove only that addition in $Q(I)$ is continuous. Proofs for all other operations and cases are similar. Thus let r, s belong to $Q(I)$, $r=\phi(a, b)$, $s=\phi(c, d)$, $r+s=t$. If U_t is a neighborhood of t and if $\phi(x, y)=t$, then one can find neighborhoods U_x, U_y such that $\phi(U_x, U_y)\subset U_t$. Since $\phi(ad+bc, bd)=t$, there are neighborhoods U_a, U_b, U_c, U_d for which $\phi(U_aU_d+U_bU_c, U_bU_d)\subset U_t$. It follows that if we set $U_r=\phi(U_a, U_b)$, $U_s=\phi(U_c, U_d)$, then $U_r+U_s\subset U_t$, and the sets U_r, U_s are open since ϕ is open. The openness of the mapping ϕ, the continuity of multiplication in I, and the fact that I is a Hausdorff space imply that $Q(I)$, as topologized, is also a Hausdorff space. (Alternately, recalling Theorem 1 and the continuity of ϕ, we see that $Q(I)$, as topologized, is a T_1-space. Since it is also a field with continuous operations, it follows that $Q(I)$ is a Hausdorff space.)

An example of a topological integral domain that is embeddable in a topological field, but for which the relation R is not open: Let I be the polynomials whose coefficient field is the rational numbers with a p-adic valuation. The valuation for a polynomial is the maximum of the valuations of the coefficients. The quotient field has a valuation given by quotients of valuations of polynomials. The polynomials are not dense in the rational functions (the function $1/x$ is at a distance of at least one from any polynomial). Therefore, by Theorem 12, below, the relation R is not open.

THEOREM 5. *If S (I) is embeddable in $Q(S)$ $(Q(I))$, then ϕ is continuous on $S\times S$ $(I\times I^*)$.*

PROOF. Consider the case of S. The proof for I is identical. Let $x=\phi(a, b)$ belong to $Q(S)$. Then for a given neighborhood of x, U_x', there are neighborhoods U_a', U_b' of a and b in $Q(S)$ such that

$U'_a (U'_b)^{-1} \subset U'_x$. (Here we have identified a and b with $\phi(a, e)$ and $\phi(b, e)$.) Let $U_a = U'_a \cap S^\sharp$, $U_b = U'_b \cap S^\sharp$. These are open sets in $S^\sharp$ and $\phi(U_a, U_b) \subset U'_x$, whence ϕ is continuous.

THEOREM 6. *If I is weakly embeddable in $Q(I)$ in the strongest topology relative to which ϕ is continuous and if $Q(I)$ in some other topology is a topological field F in which I is embedded, then I is embeddable in $Q(I)$ (in its original topology). A similar result holds for semigroups.*

PROOF. By Theorem 5, ϕ is continuous relative to the topology of F, and thus the topology of $Q(I)$ is stronger than that of F. Consequently the topology induced on $I^\sharp$ by $Q(I)$ is at least as strong as that induced on $I^\sharp$ by F. The latter topology is the topology of I, and thus the result follows, since in the former topology we have at most a weakening.

THEOREM 7. *Let F be a topological field, I and I' topological integral domains, and assume that $I \subset I' \subset F$, where the inclusions are algebraic and topological. If $F = Q(I)$, where $Q(I)$ has been provided with the strongest topology with respect to which the mapping ϕ is continuous, then $F = Q(I')$, where $Q(I')$ has been provided with the strongest topology with respect to which ϕ' (the mapping of $I' \times I'^*$ on $Q(I')$) is continuous. A similar result holds for semigroups and groups.*

PROOF. Clearly F is the algebraic quotient structure of I'. Furthermore since I' is topologically embedded in F, ϕ' is continuous (Theorem 5). We show that the topology of F is the strongest relative to which ϕ' is continuous. Otherwise, ϕ (which is the contraction of ϕ' to $I \times I^*$) would be continuous relative to a topology definitely stronger than the strongest with respect to which it is continuous, a contradiction. The proof for the case of semigroups and groups is similar.

THEOREM 8. *If $S(I)$ has the property that U open implies aU open ($a \neq 0$ in the case of an integral domain), then the relation R is open.*

PROOF. By use of Theorem 3 we have only to let $U_{ac} = cU_a$ and $U_{bc} = cU_b$ ($V = cU$).

THEOREM 9. *If $S(I)$ is algebraically a group (field) and topologically a semigroup (integral domain), the relation R is open and the topology can be weakened so that $S(I)$ becomes a topological group (field).*

PROOF. For the case I: If $a \neq 0$ and if U is open, then aU is the inverse image of the open set U under the continuous transformation $x \to a^{-1}x$, and is therefore open. By Theorem 8, the relation R is open.

Therefore the quotient field can be given a topology that makes it a topological field and that induces upon I a possibly weaker topology than the original. But I is algebraically identical with its quotient field. A similar proof holds for S.

COROLLARY. *If in the above S is compact, then S is a topological group.*

THEOREM 10. *If S (I) is a topological group (field), it is identical with $Q(S)$ ($Q(I)$), where the topology of the latter is the strongest relative to which ϕ is continuous.*

PROOF. Only the proof for I will be given. By Theorem 9, R is open, whence $Q(I)$ in the strongest topology relative to which ϕ is continuous is a topological field in which I is weakly embedded. Since in the original topology of I (which is $Q(I)$) I is actually embedded, Theorem 6 shows that I is embedded in $Q(I)$ in the first topology mentioned.

THEOREM 11. *If I is embedded densely in a topological field F, the relation R is open.*

PROOF. Let U_I be any neighborhood of zero, in I, and let U_F be an open set in F whose intersection with I is U_I. Let a, b, $c \in I$, $b \neq 0$, $c \neq 0$. Let V_F be a neighborhood of zero in F such that, for any elements v_i of V_F, $(1/c)v_1 + acv_2 + v_1v_2$ and $(1/c)v_3 + bcv_4 + v_3v_4 \in U_F$. Let $V_I = V_F \cap I$. If (x, y) is any element of $(ac + V_I, bc + V_I)$ and if z is any element of $I \cap ((1/c) + V_F)$ (the intersection is not empty), then (xz, yz) has the form $((ac + v_1)((1/c) + v_2), (bc + v_3)((1/c) + v_2))$ $= (a + (1/c)v_1 + acv_2 + v_1v_2, b + (1/c)v_3 + bcv_2 + v_3v_2) = (a + u_1, b + u_2)$, where u_1 and $u_2 \in U_F$. Since xz, yz, a, and $b \in I$, u_1 and $u_2 \in U_I$. Theorem 3 completes the proof.

THEOREM 12. *If I is a polynomial integral domain with a nondiscrete coefficient field (whose topology is induced by that of I), topologically embedded in its quotient field $F = Q(I)$ (with a topology relative to which F is a topological field), and if the relation R is open, then I is dense in F.*

PROOF. Let c be a nonzero polynomial in I and let U_F be a neighborhood of zero in F. Let V_F be a neighborhood of zero in F such that whenever x is in $1 + V_F$ and z is in V_F then $x/(c + z)$ is in $(1/c) + U_F$. Because of the openness of the relation and the fact that $(c, c)R(1, 1)$, corresponding to $V_I = I \cap V_F$, there exists a neighborhood of zero $W_I \subset V_I$ such that if w_1 and $w_2 \in W_I$, there exist elements v_1 and v_2 of V_I such that $(c + w_1, c + w_2)R(1 + v_1, 1 + v_2)$. Let λ be a nonzero constant such that λ and $2\lambda \in W_I$. Then there exist x and $y \in 1 + V_I$

such that $(c+\lambda,\ c+2\lambda)R(x,\ y)$. Therefore $(c+2\lambda)x=(c+\lambda)y$, and since $c+\lambda$ and $c+2\lambda$ are relatively prime (their difference is λ), there exists a polynomial k such that $x=k(c+\lambda)$, $y=k(c+2\lambda)$. In F, $k=x/(c+\lambda)$. But $x\in 1+V_I$ and $\lambda\in W_I\subset V_I$. Therefore $k\in(1/c)+U_F$.

THEOREM 13. *Let I_p be the integers in their p-adic topology (p a prime), and assume that $Q(I_p)$ is provided with the strongest topology with respect to which ϕ is continuous. Then* (a) *ϕ is open,* (b) *$Q(I_p)$ is a topological field,* (c) *R_p (the rationals in their p-adic topology) and $Q(I_p)$ are identical (algebraically and topologically).*

PROOF. By one of the alternative formulations of the criterion of Theorem 3, since $I_p\times I_p^*$ satisfies the first countability axiom, it suffices to show that if $a_n,\ b_n$ are null sequences in I_p and if c,f, and g are arbitrary in I_p and $cg\neq 0$, one can find null sequences $d_n,\ e_n$ such that $(f+d_n,\ g+e_n)R(cf+a_n,\ cg+b_n)$. This means that

$$e_n(cf+a_n)-d_n(cg+b_n)=fb_n-ga_n\equiv h_n.$$

Note that if r divides $cf+a_n$ and $cg+b_n$, then r divides fb_n-ga_n since $fb_n-ga_n=f(cg+b_n)-g(cf+a_n)$. Let $A_n=cf+a_n$, $B_n=cg+b_n$, $s_n=$ the greatest common divisor of A_n and B_n. There is an integer N independent of n such that for $m>N$, $p^m\nmid A_n$, $p^m\nmid B_n$. Thus if we set $P_n=A_n/s_n$, $Q_n=B_n/s_n$, there exist integers $R_n,\ S_n$ for which $R_nP_n-S_nQ_n=1$, and $h_nR_nP_n-h_nS_nQ_n=h_n$. Since $a_n,\ b_n$ are null sequences, h_n approaches 0 and hence h_n contains a factor p^{t_n}, with t_n approaching infinity. Let $s_n=k_np^{u_n}$, where k_n is prime to p. Then the numbers u_n are bounded and consequently $h_n/s_n=\phi(h_n,\ s_n)$ approaches 0, since ϕ is continuous and h_n approaches 0. Let $d_n=h_nR_n/s_n$, $e_n=h_nS_n/s_n$. Then $d_n,\ e_n$ are integers, since h_n/s_n are integers; since the u_n are bounded, it follows that d_n and e_n are null sequences, and the above computations show that they fulfill the requirements which were assigned. Thus ϕ is open, and part (a) is proved.

(b) By Theorem 2, $Q(I_p)$ is a topological field.

(c) Since I_p is embeddable in R_p, ϕ relative to R_p is continuous by Theorem 5. Thus the topology of R_p is comparable with and not strictly stronger than the topology of $Q(I_p)$. Since ϕ relative to $Q(I_p)$ is open and since $I_p\times I_p^*$ satisfies the first countability axiom, so does $Q(I_p)$. Thus to show that the topology of R_p is the same as that for $Q(I_p)$ we need only show that if r_n approaches r in R_p, then r_n approaches r in $Q(I_p)$. Since both R_p and $Q(I_p)$ are topological fields, we may assume $r=0$. Thus $r_n=(k_n/m_n)p^{t_n}$, where the t_n approach infinity and k_n and m_n are relatively prime, and prime to p. Then $\phi(k_np^{t_n},\ m_n)=r_n$ and there are integers a_n and b_n such that a_nm_n

$+b_n p^n = 1$. Thus $r_n = \phi(a_n k_n p^{i_n}, a_n m_n) = \phi(a_n k_n p^{i_n}, 1 - b_n p^n)$. Clearly $a_n k_n p^{i_n}$ is a null sequence, and $1 - b_n p^n$ approaches 1. Thus r_n approaches $\phi(0, 1) = 0$ in $Q(I_p)$, since ϕ is continuous relative to the topology of $Q(I_p)$.

COROLLARY. *If I is any integral domain algebraically and topologically contained in R_p, $Q(I)$ is R_p when $Q(I)$ is provided with the strongest topology with respect to which ϕ is continuous, and I is embeddable in $Q(I)$.*

PROOF. This is a consequence of Theorem 7.

4. Compact semigroups. In this section we temporarily abandon the conventions of notation established above.

THEOREM 14. *Let S be a semigroup with a two-sided cancellation law (we do not assume commutativity or the existence of an identity). If S is a compact Hausdorff space and if multiplication is continuous in both variables, then S is a topological group in its given topology.*[1]

PROOF. We shall need two lemmas for the proof of the above. The first is a mere exercise in algebra and its proof will not be given.

LEMMA 1. *If S is a semigroup with a two-sided cancellation law (no commutativity or existence of identity assumed) and if there are two elements a, b in S which satisfy $b = ab$, then S has a unique (two-sided) identity.*

LEMMA 2. *If G is a group and a compact Hausdorff space and if multiplication is continuous in both variables, then G is a topological group, that is, x^{-1} is a continuous function of x.*

REMARKS. In Theorem 14 and in Lemma 2, compactness is to be construed in the sense that every infinite set of S or of G has a limit point (that is, S and G are countably compact). As is well known, spaces compact in the Bourbaki-Tukey sense (Alexandroff-Urysohn: "bicompact") and sequentially compact spaces are countably compact. In what follows we shall use the notations of [5].[2]

PROOF. Assume $x(\delta) \to_\Delta x_0$. Then $(x(\delta))^{-1}$ is a set of points having a limit point y, and thus there is a function $\alpha(\delta)$ which is cofinal in Δ and such that $(x(\alpha(\delta)))^{-1} \to_\Delta y$. Continuity of multiplication now implies that $y = x_0^{-1}$. A similar argument shows that there is only one limit point of the points $(x(\delta))^{-1}$ and the continuity of division follows.

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[1] See discussion following the proof of Theorem 14.
[2] Numbers in brackets refer to the bibliography at the end of the paper.

PROOF OF THE THEOREM. Let x be an arbitrary element of S. In view of Lemma 1, we may assume that the elements x^n, $n = 1, 2, \cdots$, are distinct. Thus they have a limit point z_0, and there is a function $n(\delta)$ such that $x^{n(\delta)} \to_\Delta z_0$. Clearly we may assume that all $n(\delta)$ are greater than 1. Thus the points $x^{n(\delta)-1}$ have a limit point z_1, and there is a function $\alpha(\delta)$ cofinal in Δ for which $x^{n(\alpha(\delta))-1} \to_\Delta z_1$, whence $xz_1 = z_0$, owing to the continuity of multiplication. In this manner one constructs a sequence z_n such that $x^n z_n = z_0$ (the z_n will be distinct!). There is a function $\beta(\delta)$ which is cofinal in Δ and for which $z_{n(\beta(\delta))} \to_\Delta z'$, where z' is a limit point of the sequence z_n. Hence $x^{n(\beta(\delta))} z_{n(\beta(\delta))} \to_\Delta z_0 z'$ $= z_0$, and we conclude, using Lemma 1, that S has an identity.

Since xS is again a semigroup satisfying the hypotheses of Theorem 14, it follows that xS has an identity which is easily seen to be *the* identity of S. Thus x^{-1} exists and S is a group. An application of Lemma 2 yields the desired result.

Note that the Hausdorff property of the space is employed only to insure the uniqueness of limits. If we relax this restriction and demand only that points be closed sets (T_1), the theorem fails to hold as the following example shows: Let N be the non-negative natural numbers, clearly an additive (indeed abelian) semigroup with a cancellation law. Topologize N by defining the neighborhoods of a point n as follows: a neighborhood of n consists of the point n and the complement of some finite set of integers. Thus topologized, N is a compact T_1 space which is not a Hausdorff space, and addition is continuous in both variables.

Iwasawa has published a proof of Theorem 14 in Sūgaku vol. 1 (1947), for the case of S bicompact. At the suggestion of the editors we include a brief sketch of his proof: By Zorn's lemma there is a minimal nonvacuous closed subset A of S such that $AS \subset A$. Thus for a in A, $aS = A$. If x is in S, a in A, then ax is in A, $axS = A = aS$, and $xS = S$. Dually, $Sx = S$, whence S is a group. The class of all left and right translations of S is equicontinuous since multiplication is continuous on the compact space $S \times S$ and therefore uniformly continuous on $S \times S$. The continuity of inversion is shown as follows: if x is close to y, then xy^{-1} is close to the identity and y^{-1} is close to x^{-1}. According to a personal communication, J. E. L. Peck has obtained a similar result.

5. **Metric semigroups.** If S is a semigroup with an *invariant metric*, then S is a topological semigroup since $a_n \to a$, $b_n \to b$ imply $d(a_n b_n, ab)$ $\leq d(a_n b_n, a_n b) + d(a_n b, ab) = d(b_n, b) + d(a_n, a) \to 0$, $a_n b_n \to ab$, where d is the distance in S. We now introduce a norm in $Q(S)$ as follows: if $A = \phi(a, b)$ is in $Q(S)$, we define

$$\|A\| = \inf_{(a', b')R(a, b)} (d(a', e) + d(b', e)).$$

Regarding this norm the following theorem holds.

THEOREM 15. *Let S be as above and let $Q(S)$ be metrized by $D(A, B)$ $= \|AB^{-1}\|$. Then $Q(S)$ is a topological group with an invariant metric D; $Q(S)$ is discrete if and only if S is discrete; ϕ is open on $S \times S$; ϕ restricted to $S \times e$ is open and continuous at e; finally, if R is open, ϕ restricted to $S \times e$ is continuous and thus S is embeddable in a topological group with an invariant metric.*

PROOF. 1. $\| \cdot \|$ is a norm of $Q(S)$: Clearly $\|A\| \geqq 0$ and $A = E$, the identity of $Q(S)$, implies $\|A\| = 0$. If $\|A\| = \|\phi(a, b)\| = 0$, there exist sequences a_n, b_n of elements of S such that $(a_n, b_n)R(a, b)$, $a_n \to e$, $b_n \to e$, whence, since $a_n b = a b_n$, $b = a$ and $A = E$. We next observe that $\|AB\| \leqq \|A\| + \|B\|$. For let $A = \phi(a_1, a_2)$ and $B = \phi(b_1, b_2)$, AB $= \phi(a_1 b_1, a_2 b_2)$. Given any positive ϵ, select a_1', b_1', a_2', b_2' such that $(a_1', a_2')R(a_1, a_2)$, $(b_1', b_2')R(b_1, b_2)$ and $d(a_1', e) + d(a_2', e) \leqq \|A\| + \epsilon$, $d(b_1', e) + d(b_2', e) \leqq \|B\| + \epsilon$. Now $\|AB\| \leqq d(a_1' b_1', e) + d(a_2' b_2', e)$ since $(a_1' b_1', a_2' b_2')R(a_1 b_1, a_2 b_2)$. But $d(a_1' b_1', e) + d(a_2' b_2', e) \leqq d(a_1' b_1', b_1')$ $+ d(b_1', e) + d(a_2' b_2', b_2') + d(b_2', e) = d(a_1', e) + d(b_1', e) + d(a_2', e)$ $+ d(b_2', e) \leqq \|A\| + \|B\| + 2\epsilon$ and $\|AB\| \leqq \|A\| + \|B\|$ as promised. Note further that $\|A\| = \|A^{-1}\|$: $\|A\| = \|\phi(a, b)\| = \inf_{(a', b')R(a, b)}(d(a', e) + d(b', e)) = \inf_{(b', a')R(b, a)}(d(b', e) + d(a', e)) = \|\phi(b, a)\| = \|A^{-1}\|$. If now we define $D(A, B) = \|AB^{-1}\|$, we see that D is an invariant metric of $Q(S)$; the continuity of multiplication and division is a consequence of the invariance of the metric.

2. ϕ restricted to $S \times e$ is continuous at the identity: if $A = \phi(a, e)$, $D(A, E) = \|A\| = \inf_{(a', b')R(a, e)}(d(a', e) + d(b', e)) \leqq d(a, e)$. Therefore, $a_n \to e$ implies that $A_n \to E$.

3. ϕ is open. Let $A_n = \phi(a_n, b_n) \to \phi(a, b) = A$. We wish to prove that there exist sequences a_n', b_n' of elements of S such that $(a_n', b_n')R(a_n, b_n)$ and $(a_n', b_n') \to (a, b)$. Now $\inf_{(a', b')R(a_n b, a b_n)}(d(a', e) + d(b', e)) \to 0$ implies that there exist sequences a_n'', b_n'' of elements of S such that $d(a_n'', e) \to 0$, $d(b_n'', e) \to 0$, $(a_n'', b_n'')R(a_n b, a b_n)$ or $a_n'' a b_n = a_n b b_n''$; if we let $a_n' = a_n'' a$, $b_n' = b_n'' b$, our assertion is proved.

4. ϕ restricted to $S \times e$ is open. Let $A_n = \phi(a_n, e) \to \phi(a, e)$ $= A$; to show that $a_n \to a$. We see that $\|A_n A^{-1}\| = \|\phi(a_n, a)\|$ $= \inf_{(a', b')R(a_n, a)}(d(a', e) + d(b', e)) \to 0$. Hence, as above, there exist sequences a_n', b_n' of elements of S such that $a_n' \to e$, $b_n' \to e$, $a_n' a = a_n b_n'$. Finally $d(a_n, a) = d(a_n b_n', a b_n') = d(a_n' a, a b_n') = d(a_n', b_n') \leqq d(a_n', e) + d(e, b_n') \to 0$, and our assertion is proved. The fact that $Q(S)$ is discrete if and only if S is discrete is a consequence of 2 and 4.

5. Let now R be open; we wish to show that ϕ contracted to $S \times e$ is continuous, or that $a_n \to a$ implies that $A_n = \phi(a_n, e) \to A = \phi(a, e)$. But $D(A_n, A) = \|A_n A^{-1}\| = \inf_{(a', b')R(a_n, a)}(d(a', e) + d(b', e))$. Since $(a_n, a) \to (a, a)$ and $(a, a)R(e, e)$, we have the existence of sequences a_n', b_n' of elements of S such that $(a_n', b_n')R(a_n, a)$, $(a_n', b_n') \to (e, e)$ because R is open and $S \times S$ and $Q(S)$ are metric (see paragraph immediately preceding Theorem 4). This last relation shows that $D(A_n, A) = \inf_{(a', b')R(a_n, a)}(d(a', e) + d(b', e)) \leqq d(a_n', e) + d(b_n', e) \to 0$ as promised.

THEOREM 16. *Let S be a metric topological semigroup such that R is open. Then the property*

(1) $a_n b_n \to b, \ a_n \to e \ implies \ b_n \to b$

is a necessary and sufficient condition for embeddability.

PROOF. The necessity is obvious. The sufficiency is proved as follows. In the strongest topology on $Q(S)$ relative to which ϕ is continuous, ϕ is an open continuous mapping by Theorem 2. ϕ contracted to $S \times e$ is continuous. Let now $y_n \to y$, where y_n and y are in $S^{\sharp}$, $y_n = \phi(x_n, e)$ and $y = \phi(x, e)$. Since ϕ is open there exist sequences b_n, c_n of elements of S such that $y_n = \phi(b_n, c_n)$, $(b_n, c_n) \to (x, e)$; hence $(b_n, c_n)R(x_n, e)$ or $b_n = c_n x_n$ where $c_n \to e$, $b_n \to x$, hence, by (1), $x_n \to x$; ϕ contracted to $S \times e$ is open, whence our assertion.

Note that since an invariant metric implies (1), the last part of Theorem 15 is a consequence of the preceding theorem. We also note the following consequences of the foregoing: if S as described in our theorem is (algebraically) a group, it is a topological group. For future reference we observe that the following is a necessary and sufficient condition for embeddability which is a stronger necessary condition than (1): if $a_n b_n \to c$, and $a_n \to a$, then b_n converges.

6. **Complete separable metric semigroups.** Except when the contrary is indicated all semigroups S mentioned hereinafter are *complete separable metric topological semigroups* with the additional property:

(2) *If U is an open subset of S, then so is aU for all a in S* (cf. Theorem 8).

LEMMA 3. *Let $\{U\}$ be a basis of neighborhoods of e. Then $\{aU\}$ is a basis of neighborhoods of a. If f is a homomorphism of S onto $S' = f(S)$ which is continuous at e, then it is continuous.*

PROOF. This lemma uses only property (2) and the continuity of ab in each variable. 1. Since aU is open and since every open set contain-

ing a contains some aU (by one-sided continuity of ab), the first assertion is clear. 2. Let now U' be a neighborhood of $e' = f(e)$. There exists a neighborhood V of e such that $f(V) = V'$ is contained in U'. Consider now a' in S' and $a'U'$; since V' is in U', $a'V' = (aV)'$ is in $a'U'$ which proves the second assertion.

NOTE. If S is any topological semigroup and $\{U\}$ a basis of neighborhoods of e, then if we retopologize S by letting $\{aU\}$ be a basis at a, we get a stronger topology of S with respect to which S is still a topological semigroup. For, by continuity of multiplication, every open set containing a contains some aU, and for each au in aU there exists U_1 such that auU_1 is in aU: for u in U there exists U_1 such that uU_1 is in U and, therefore, auU_1 is in aU. The new topology of S has at least as strong separation properties (T_0, T_1, T_2) as the old one, and, of course, satisfies condition (2) above. Now a semigroup with property (2) automatically has one-sided continuity of multiplication: $a(bU)$ is in $(ab)U$; and the simultaneous continuity of multiplication in S in the stronger topology is a consequence of the same property at e in the original topology: given U there exists U_1 such that U_1^2 is in U; hence $aU_1bU_1 = abU_1^2$ is in abU.

LEMMA 4. *If C_i is a subset of S of the ith category ($i = 1, 2$), then so is aC_i for all a in S; and conversely, if aC is of the ith category, then so is C.*

PROOF. 1. Let the subset N of S be nowhere dense in S; we prove that aN is nowhere dense in S. We first observe that S and aS are homeomorphic, because $S \to aS$ is open by hypothesis, continuous by one-sided continuity of multiplication, and 1-1 by the cancellation law. (Note, incidentally, that the homeomorphism of S and aS, although implied by condition (2), is weaker than (2), as is shown by the example of the additive semigroup of the nonnegative reals with the usual topology.) We observe that if $(aX)^{cc}$ denotes the relative closure of aX in aS, then $(aX)^{cc} = aX^c$ (where the single superscript c denotes the closure in S): $(aX)^{cc} = (aX)^c \cap aS$ since y in $(aX)^{cc}$ is equivalent to y's being in $(aX)^c$ and in aS. Now, $(aX)^c \cap aS = aX^c$. We know by continuity of multiplication that aX^c is a subset of $(aX)^c$; hence $aX^c \cap aS = aX^c$ is in $(aX)^c \cap aS$. Let now $z = ay$ be in $(aX)^c \cap aS$. Since ayU is open, where U is a neighborhood of e, $ayU \cap aX$ is not empty; therefore, $yU \cap X$ is not empty, since $\{yU\}$ is a basis of neighborhoods of y by the first part of Lemma 3. We can conclude that y is in X^c, $z = ay$ is in aX^c and, therefore, $(aX)^c \cap aS$ is a subset of aX^c, $(aX)^{cc} = (aX)^c \cap aS = aX^c$. Finally, in order to prove that aN is nowhere dense, assume, on the contrary, that there

exists an open set G such that G is dense in $(aN)^c$. Then $G \cap aS$ is not empty since g in G implies that g is in $(aN)^c$; therefore, $G \cap aN$ is not empty, and $G \cap aS$ is not empty. Now $G \cap aS$ is a subset of $(aN)^c \cap aS = aN^c$. But since N is nowhere dense in S, aN is nowhere dense in aS (by 1 above); hence $(aN)^{cc}$ contains no set open in aS; but $(aN)^{cc} = aN^c$ and $G \cap aS$ is open in aS, a contradiction. We now can conclude that aC_1 is of the first category in S if C_1 is of the first category in S.

2. Let aN be nowhere dense in S; we wish to prove that N is nowhere dense in S. Assume, on the contrary, that there exists an open set G in N^c; then the open set aG is in aN^c which in turn is contained in $(aN)^c$ and aN is not nowhere dense, a contradiction. Again we can conclude that if aC is of the first category in S, so is C.

3. The statements concerning C_2, aC_2, and the converse follow.

LEMMA 5. *Let the semigroup S have the property*

$$(3) \qquad \text{if } a_n b_n \to e, \ b_n \to e, \ \text{then } a_n \to e.$$

Then, if $x_n y_n \to y$ and $y_n \to y$, $x_n \to e$.

PROOF. Since yU is open for open U, $x_n y_n$ and y_n are in yU for sufficiently large n; therefore, $x_n y_n = y e_n$, $y_n = y e_n'$ where $e_n \to e$, $e_n' \to e$; hence $x_n y_n = x_n y e_n' = y e_n$, $x_n e_n' = e_n$ and $x_n \to e$ by (3), as desired.

THEOREM 17. *Let S_1 be a semigroup with property* (3), *and S_2 another semigroup. Then every continuous isomorphism of S_1 onto S_2 is open.*

PROOF. By Lemma 3 it is sufficient to prove that the mapping f of S_1 onto S_2 is open at the identity. Let then $f(x_n) = y_n \to e_2 = f(e_1)$. We wish to prove that $x_n \to e_1$. Let ϵ_i be a sequence of positive real numbers such that $\epsilon_i \to 0$. Since S_1 is separable it may be covered by countably many ϵ_1-spheres. Their images cover S_2 which, since it is complete, is of the second category. Therefore, one of these ϵ_1-spheres, say T_1, is such that $f(T_1)$ is of the second category. Since T_1 is a Borel set, $f(T_1)$ is a Suslin set, and, therefore, a Baire set (see [3, p. 249]). Thus there exists a nonempty open set L_1 in S_2 such that $L_1 \cap Cf(T_1)$ is of the first category where C denotes the complement; and there exists a nonempty open set L_{11} in L_1 such that $f(T_1)$ is of the second category at each point of L_{11}, while $L_{11} \cap Cf(T_1)$ is of the first category. Consider now any y in L_{11}. Owing to the continuity of multiplication in S_2, there exist open sets L_{12}, L_{13} such that y is in L_{12} which in turn is a subset of L_{11}, e_2 is in L_{13}, and $L_{12}L_{13}$ is a subset of L_{11}. Since $y_n \to e_2$, there exists a natural number n_1 such that, for $n \geq n_1$, y_n is in L_{13}, or $y_n L_{12}$ is a subset of L_{11}. Note that $f(T_1)$ is of the second cate-

gory at each point of L_{12} and that $L_{12} \cap Cf(T_1)$ is of the first category. Consider now $L_{1n} = y_n(L_{12} \cap f(T_1))$ which is of the second category by our above arguments and by Lemma 4. The equation

$$L_{1n} = (L_{1n} \cap f(T_1)) \cup (L_{1n} \cap Cf(T_1))$$

shows that $L_{1n} \cap f(T_1)$ is of the second category, hence not empty, because L_{1n} is a subset of $y_n L_{12}$ which in turn is a subset of L_{11}, and $L_{11} \cap Cf(T_1)$ and hence $L_{1n} \cap Cf(T_1)$ are of the first category. Select now $y_{1,n}$ in $L_{1n} \cap f(T_1)$. $y_{1,n}$ is in $f(T_1)$ or $y_{1,n} = f(x_{1,n})$ for some $x_{1,n}$ in T_1; but $y_{1,n}$ is also in L_{1n} which is a subset of $y_n f(T_1)$, or $y_{1,n} = y_n f(x_{2,n})$ for some $x_{2,n}$ in T_1, $x_{1,n} = x_n x_{2,n}$ for $n \geq n_1$.

Assume now that we have a sequence of open spheres in S_1: $T_1 \supset T_2 \supset \cdots \supset T_m$, T_i of radius less than or equal to ϵ_i, such that $f(T_i)$ is of the second category; a sequence of natural numbers $0 < n_1 < n_2 < \cdots < n_m$; sequences $x_{2,k}$, $x_{1,k} = x_k x_{2,k}$ for $n_1 \leq k < n_m$ where, if $n_i \leq k < n_{i+1}$, $x_{2,k}$ and $x_{1,k}$ are in T_i. Assume further that for $k \geq n_m$, there exist sequences $x'_{2,k}$, $x'_{1,k} = x_k x'_{2,k}$ such that $x'_{2,k}$ and $x'_{1,k}$ are in T_m. We now cover T_m by a countable number of open spheres of radii less than or equal ϵ_{m+1} each of which is contained in T_m. Since $f(T_m)$ is of the second category, one of these spheres, say T_{m+1}, will have an image $f(T_{m+1})$ of the second category. Just as in the first part of the proof, we conclude that there exists a nonempty open set $L_{m+1,1}$ contained in S_2 such that $f(T_{m+1})$ is of the second category at each point of $L_{m+1,1}$ while $L_{m+1,1} \cap Cf(T_{m+1})$ is of the first category; that there exists a natural number $n_{m+1} > n_m$ and a nonempty open set $L_{m+1,2}$ contained in $L_{m+1,1}$ such that, for $n \geq n_{m+1}$, $y_n L_{m+1,2}$ is contained in $L_{m+1,1}$; and finally that, for $n \geq n_{m+1}$, there exist sequences $x'_{2,n}$, $x'_{1,n} = x_n x'_{2,n}$ such that $x'_{2,n}$ and $x'_{1,n}$ are in T_{m+1}.

We now define $x_{2,k}$ and $x_{1,k} = x_k x_{2,k}$ for $n_m \leq k < n_{m+1}$ to be equal to $x'_{2,k}$ and $x'_{1,k}$ respectively. This completes our induction. If we set $x_{2,k} = e_1$ for $0 < k < n_1$, and $x_{1,k} = x_k x_{2,k}$, we have defined sequences n_i, $x_{2,n}$, $x_{1,n} = x_n x_{2,n}$ for all n, such that $0 < n_1 < \cdots$ and, if $n_i \leq n < n_{i+1}$, $x_{2,n}$ and $x_{1,n}$ are in T_i and, therefore, are Cauchy sequences. Owing to the completeness of S_1 there exists x_2 such that $x_{2,n} \to x_2$ and also $x_{1,n} \to x_2$, since these two Cauchy sequences are concurrent. Our hypotheses and Lemma 5 now imply that $x_n \to e_1$ since we have $x_n x_{2,n} \to x_2$, $x_{2,n} \to x_2$. This concludes our proof.

Note that if S_1 has the property

(4) *there exists a neighborhood of the identity all of whose elements have inverses, and division where possible is continuous,*
then condition (3) is satisfied. Note also that if the metric of S_1 is invariant, (3) is true. (The invariance of the metric of S_1 would ma-

terially simplify the proof of the above theorem.)

The theorem just proved is a generalization of a theorem of Banach, cf. [1, Theorem 8; 4, Theorem 13].

It is clear that the foregoing theorem will be of use in resolving questions of embeddability of semigroups satisfying the hypotheses for S_1 if the following are satisfied:

$Q(S)$ is complete (that it is metric is clear); $S^\sharp$ is closed in $Q(S)$; $AS^\sharp$ is open in $S^\sharp$ for all A in $S^\sharp$. This last condition implies that if $U^\sharp$ is open in $S^\sharp$, then so is $AU^\sharp$ for all A in $S^\sharp$, for $U^\sharp = S^\sharp \cap U$ for some open subset U of $Q(S_1)$, and $AU^\sharp = A(S^\sharp \cap U) = AS^\sharp \cap AU$ which is open since $Q(S_1)$ is a group (see Theorem 2).

LEMMA 6. *$S^\sharp$ is closed in $Q(S)$ if and only if bS is closed in S for all b in S.*

PROOF. We note that bS is closed if and only if $a_n \to a$, $b | a_n$ imply $b | a$ (where $x | y$ means that there exists z such that $y = zx$)—in other words that the relation $|$ is topological in character. We first note that $b | a$ is equivalent to a's belonging to bS. Let now bS be closed, $a_n \to a$, $b | a_n$, that is, a_n in bS. Then a is in bS and $b | a$. Let finally $a_n \to a$, $b | a_n$ imply that $b | a$ and consider bS. Let $a_n \to a$ where a_n is in bS; we wish to show that a is also in bS. But a_n in bS means that $b | a_n$; hence $b | a$, a is in bS as desired. Let now bS be closed; we wish to show that $S^\sharp$ is closed, that is, if $A_n = \phi(a_n, e) \to A = \phi(a, b)$, then A is in $S^\sharp$, that is, $b | a$. But $\phi(a_n, e) \to \phi(a, b)$ implies the existence of a_n', b_n' such that $a_n' \to a$, $b_n' \to b$, $a_n' = a_n b_n'$; $b_n' = b e_n$ (where $e_n \to e$), $a_n' = a_n b e_n$, or $b | a_n'$; hence $b | a$ and $S^\sharp$ is closed. Lastly, let $S^\sharp$ be closed. We wish to prove that $b | a_n$, $a_n \to a$ imply $b | a$ which, by the first part of the present proof, is equivalent to showing that bS is closed. But $b | a_n$ means that there exists a_n' such that $a_n = a_n' b$. Thus we have $\phi(a_n, b) \to \phi(a, b)$; but $\phi(a_n, b) = \phi(a_n', e)$; therefore, $\phi(a, b)$ is in $S^\sharp$ as desired.

Note that the fact that bS is closed for b in S implies that bF is closed for all closed subsets F of S and all b in S: $bf_n \to x$ implies that $b | x$ or that x is in bS, or that $x = bs$; hence we have $bf_n \to bs$ or $bf_n = bse_n$ (where $e_n \to e$), $f_n = se_n \to s$ in F, $x = bs$ in bS, bF is closed.

LEMMA 7. *Condition (4) implies that bS is closed for all b in S.*

PROOF. Observe that $b | a_n$, $a_n \to a$ imply that $a_n = a_n' b$ for some a_n', and $a_n = ae_n$ for $e_n \to e$; therefore, $ae_n = a_n' b$, for n sufficiently large, $a = a_n' b e_n^{-1}$ and $b | a$.

We mention two further conditions insuring that bS be closed for all b in S: (1) bS is closed if and only if $bs_n \to x$ implies that there

exists s such that $s_n \to s$; (2) if the metric of S is invariant, or (which is implied by it) if the mapping $bS \to S$ is uniformly continuous, then bS is closed for all b in S.

LEMMA 8. *Condition* (4) *implies that* $AS^{\#}$ *is open in* $S^{\#}$ *for all* A *in* $S^{\#}$.

PROOF. Note first of all that by an earlier remark this implies that if $U^{\#}$ is open in $S^{\#}$ and if A is an element of $S^{\#}$, then $AU^{\#}$ is open in $S^{\#}$. Let $A_n = \phi(a_n, e) \to A_0 B_0 = \phi(a_0 b_0, e)$. We wish to prove that $A_n = A_0 A_n'$ for A_n' in $S^{\#}$ and for n large enough. Our hypothesis implies the existence of a_n', b_n' such that $a_n' \to a_0 b_0$, $b_n' \to e$, $a_n' = a_n b_n'$; but $(b_n')^{-1}$ exists for n large enough; hence $a_n'(b_n')^{-1} = a_n$ and $a_n' = a_0 b_0 e_n$ (where $e_n \to e$) or $a_0 b_0 e_n (b_n')^{-1} = a_n = a_0 a_{1,n}$, where $a_{1,n} = b_0 e_n (b_n')^{-1}$, $A_n = \phi(a_n, e) = \phi(a_0 a_{1n}, e) = A_0 A_n'$ where $A_n' = \phi(a_{1,n}, e)$, as desired. A similar proof shows that (4) also implies that $S^{\#}$ is open in $Q(S)$.

These lemmas establish the truth of the following corollary to Theorem 17:

COROLLARY. *If* S *is a locally compact semigroup satisfying condition* (4), *then* S *is embeddable in a locally compact group.*

PROOF. We note that $Q(S)$ is the continuous open image of the locally compact separable metric semigroup $S \times S$; hence it itself is locally compact and separable metric; it is complete because it is a topological group which is locally compact.

The hypotheses of the preceding corollary are not strong enough to make S a group; a counterexample is the multiplicative semigroup of the nonzero p-adic integers; nor does condition (4) imply condition (2)—witness the multiplicative semigroup of p-adic numbers of absolute value greater than or equal to 2, together with 1 (which is then an isolated point).

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