A NONABSOLUTE BASIS FOR HILBERT SPACE

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Al'tman [1][1] exhibits the following biorthonormal system $\{f_n; g_n\}$ over $(-\pi, \pi)$,

$$f_0 = \frac{|x|^\alpha}{(2\pi)^{1/2}}, \quad f_1 = \frac{|x|^\alpha}{(\pi)^{1/2}} \cos x, \quad f_2 = \frac{|x|^\alpha}{(\pi)^{1/2}} \sin x, \cdots,$$

$$f_{2m+1} = \frac{|x|^\alpha \cos (m+1)x}{(\pi)^{1/2}}, \qquad f_{2m+2} = \frac{|x|^{-\alpha} \sin (m+1)x}{(\pi)^{1/2}},$$

$$m = 0, 1, 2, \cdots,$$

and

$$g_0 = \frac{|x|^{-\alpha}}{(2\pi)^{1/2}}, \quad g_1 = \frac{|x|^{-\alpha} \cos x}{(\pi)^{1/2}}, \quad g_2 = \frac{|x|^\alpha \sin x}{(\pi)^{1/2}}, \cdots,$$

$$g_{2m+1} = \frac{|x|^{-\alpha} \cos (m+1)x}{(\pi)^{1/2}}, \qquad g_{2m+2} = \frac{|x|^{\alpha} \sin (m+1)x}{(\pi)^{1/2}},$$

$$m = 0, 1, 2, \cdots,$$

$0<\alpha<1/2$, which is a basis for $L_2(-\pi, \pi)$, but which is neither a Bessel nor a Hilbert basis, that is:

1. There is a $y \in L_2(-\pi, \pi)$ such that $y = \sum_{n=0}^{\infty} (y, g_n) f_n$, and $\sum_{n=0}^{\infty} |(y, g_n)|^2 = \infty$.

2. There is a sequence a_n such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ and yet there is no $z \in L_2(-\pi, \pi)$ such that $z = \sum_{n=0}^{\infty} a_n f_n$, that is, such that $(z, g_n) = a_n$ for all n.

We shall show that $\{f_n; g_n\}$ is not an absolute basis for $L_2(-\pi, \pi)$ (see [2, p. 188]).

LEMMA.

$$\liminf_{n\to\infty} \|f_n\|_2 > 0.$$

PROOF.

$$\|f_{2m+1}\|_2^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x|^{2\alpha} \cos^2 (m+1) x dx,$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$\|f_{2m+2}\|_2^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|^{-2\alpha} \sin^2 (m+1)x dx, \qquad m = 0, 1, 2, \cdots.$$

Thus

$$\|f_{2m+1}\|_2^2 = \frac{1}{\pi}\left[\frac{1}{2}\int_{-\pi}^{\pi} |x|^{2\alpha}dx + \frac{1}{2}\int_{-\pi}^{\pi} |x|^{2\alpha} \cos 2(m+1)xdx\right],$$

$$\|f_{2m+2}\|_2^2 = \frac{1}{\pi}\left[\frac{1}{2}\int_{-\pi}^{\pi} |x|^{-2\alpha}dx - \frac{1}{2}\int_{-\pi}^{\pi} |x|^{-2\alpha} \cos 2(m+1)xdx\right].$$

As a consequence of the Riemann-Lebesgue lemma, the second integral of each line approaches 0 as m approaches infinity since both $|x|^{2\alpha}$ and $|x|^{-2\alpha}$ belong to $L_1(-\pi, \pi)$. The first integral of the first line has the value $(2/(2\alpha+1))\pi^{2\alpha+1}$, and the first integral of the second line has the value $(2/(1-2\alpha))\pi^{1-2\alpha}$. Thus the assertion of the lemma follows.

THEOREM. $\{f_n; g_n\}$ *is a nonabsolute basis for* $L_2(-\pi, \pi)$.

REMARK. The proof which follows applies to real Hilbert space. However, the following statement is easily verified and shows that the assertion of the theorem is valid in complex Hilbert space as well: If $\{f_n; g_n\}$ is a basis [an absolute basis] for real $L_2(-\pi, \pi)$, then it is also a basis [an absolute basis] for complex $L_2(-\pi, \pi)$.

PROOF OF THEOREM. Let $x_n = f_n/\|f_n\|_2$, $X_n = \|f_n\|_2 g_n$. Then if $\{f_n; g_n\}$ is an absolute basis, so is $\{x_n; X_n\}$, and furthermore, $\|x_n\|_2 \equiv 1$. Thus using [2, Theorem 14],[2] we see that, for each $y \in L_2(-\pi, \pi)$, $\sum_{n=0}^{\infty} |(y, X_n)|^2 < \infty$. For the special y of 1., $\sum_{n=0}^{\infty} |(y, g_n)|^2 = \infty$. Now $\sum_{n=0}^{\infty} |(y, X_n)|^2 = \sum_{n=0}^{\infty} |(y, g_n)|^2 \|f_n\|_2^2$, whence $\lim \inf_{n\to\infty} \|f_n\|_2^2 = 0$. Since this contradicts the lemma, the assertion follows.

BIBLIOGRAPHY

1. M. S. Al'tman, Doklady Akademii Nauk SSSR (N.S.) vol. 69 (1949) pp. 483–485 (Russian).

2. B. R. Gelbaum, *Expansions in Banach spaces*, Duke Math. J. vol. 17 (1950) pp. 187–196.

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[2] For $18 \geqq n \geqq 14$, Theorem n of [2] should be numbered Theorem $(n-4)$. In line 12, p. 193, read "Lemma 1 shows that . . . " for "Lemma 2 shows that . . . ." The references of this note are given in terms of the original numeration of [2].