SELF-ADJOINT FACTORIZATIONS OF DIFFERENTIAL OPERATORS

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In this short paper we prove the following result:

THEOREM. *Let L be an ordinary linear differential operator*

$$L = p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x)\cdot$$

of even order $n=2r$. $p_i(x) \in C^{n-i}$ *and* $p_0(x) > 0$ *in some closed finite interval* $[a, b]$. *Then there exists a subinterval of* $[a, b]$ *in which L has a factorization*

$$L = f(x)P_1P_2 \cdots P_r$$

where each P_α *is of the second order and formally self-adjoint.*

The theorem follows by complete induction after the proofs of Lemmas 1 and 2 below. We use the following notation: If M is a linear differential operator, then its formal or Lagrange adjoint will be denoted by M^+.

LEMMA 1. *Let*

$$N = \frac{d^n}{dx^n} + q_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + q_n(x)\cdot$$

be a linear differential operator with $q_i(x) \in C^0$ *in some closed finite interval* $[a, b]$. *Then there is a subinterval* $[a', b']$ *of* $[a, b]$ *in which N has the representation*

$$N = PM$$

where $P = P^+$ *is of second order.*

PROOF. Let $\{\phi_1(x), \phi_2(x), \cdots, \phi_n(x)\}$ be n linearly independent solutions of $Nu=0$ with Wronskian $W(x)$. There exist $n-2$ functions among the $\phi_1, \phi_2, \cdots, \phi_n$ whose Wronskian $\omega(x)$ is not identically zero in some subinterval of $[a, b]$. Let these $n-2$ functions be $\phi_1(x)$, $\phi_2(x), \cdots, \phi_{n-2}(x)$ and let $\omega(x)$ be unequal to zero in $[a', b']$.

Define the operator M by the equation:

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$$Mu = \frac{+1}{W(x)} \begin{vmatrix} \phi_1(x) & \phi_2(x) & \cdots & \phi_{n-2}(x) & u \\ \phi_1'(x) & \phi_2'(x) & \cdots & \phi_{n-2}'(x) & u' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_1^{(n-2)}(x) & \phi_2^{(n-2)}(x) & \cdots & \phi_{n-2}^{(n-2)}(x) & u^{(n-2)} \end{vmatrix}$$

$$\equiv s_2(x)u^{(n-2)} + s_3(x)u^{(n-3)} + \cdots + s_n(x)u.$$

Let

$$P = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)\cdot$$

be so chosen that $PM = N$. Now

$$PMu = as_2u^{(n)} + [a(2s_2' + s_3) + bs_2]u^{(n-1)} + \cdots,$$
$$Nu = u^{(n)} + q_1u^{(n-1)} + \cdots.$$

Comparing coefficients and noting that $(Ws_2)' + (Ws_3) = 0$, we see that $a'(x) = b(x)$ and hence that $P = P^+$.

LEMMA 2. *Let*

$$L = p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x)\cdot$$

be a linear differential operator with $p_i(x) \in C^{n-i}$, $p_0(x) > 0$ *in some closed finite interval* $[a, b]$. *Then there is subinterval* $[a', b']$ *of* $[a, b]$ *such that L has a representation*

$$L = SQ$$

where $Q = Q^+$ *is of second order.*

PROOF. Let $N = (1/p_0(x))L$. Then N is a linear differential operator with leading coefficient 1. Hence N^+ has leading coefficient 1. By Lemma 1, there exists a subinterval of $[a, b]$ such that

$$N^+ = QR$$

with $Q = Q^+$. Taking adjoints of the above equation:

$$N = R^+Q^+ = R^+Q.$$

Now

$$L = p_0(x)N = p_0(x)R^+Q.$$

Let $p_0(x)R^+ = S$. Then $L = SQ$.

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