THE WEDDERBURN PRINCIPAL THEOREM IN BANACH ALGEBRAS

CHESTER FELDMAN

The Principal Theorem of Wedderburn for a finite-dimensional algebra A states that A is the vector space direct sum of its radical R and an algebra isomorphic to A/R. It will be shown that the corresponding theorem is not true for all Banach algebras, but that it is true with certain restrictions.

The terminology of Jacobson [3][1] will be followed for radical, quasi-inverse, and quasi-regular. The notations $x \circ y = x+y+xy$ and x' for the quasi-inverse of x will also be employed.

DEFINITION 1. A Banach algebra is a complete normed linear space which is also an algebra over the complex numbers satisfying $\|xy\| \leqq \|x\| \, \|y\|$.

All the following results are proved for real algebras in [1] by the same methods.

To show that the Wedderburn theorem does not hold for an arbitrary Banach algebra, consider the commutative algebra A which is the completion of the algebra of all finite sums

$$\sum_{i=1}^{n} \alpha_i e_i + \beta r$$

where α_i and β are complex, e_i are mutually orthogonal idempotents, $r^2=0$, $e_i r = r e_i = 0$, and

$$\left\| \sum \alpha_i e_i + \beta r \right\| = \max \left\{ \left[\sum | \alpha_i |^2\right]^{1/2}, \ \left| \beta - \sum \alpha_i \right| \right\}.$$

It is easy to show this defines a norm, but it is also necessary to verify that $\|xy\| \leqq \|x\| \, \|y\|$. Let $x = \sum \alpha_i e_i + \gamma r$, $y = \sum \beta_i e_i + \nu r$. Then $xy = \sum \alpha_i \beta_i e_i$; $\|xy\| = \max \left\{ \left[\sum |\alpha_i \beta_i|^2\right]^{1/2}, \left[\sum \alpha_i \beta_i\right| \right\}$. By the Cauchy inequality,

$$\left| \sum \alpha_i \beta_i \right| \leqq \sum | \alpha_i \beta_i | \leqq \left[\sum | \alpha_i |^2\right]^{1/2} \left[\sum | \beta_i |^2\right]^{1/2}.$$

Together with $\sum |\alpha_i \beta_i|^2 \leqq \sum |\alpha_i|^2 \sum |\beta_i|^2$ this shows $\|xy\| \leqq \|x\| \, \|y\|$. Hence A is a Banach algebra.

A/R is the algebra of all sequences $\sum \alpha_i u_i$ where $u_i^2 = u_i$, α_i are complex, and $\left\| \sum \alpha_i u_i \right\| = \left[\sum |\alpha_i|^2\right]^{1/2} < \infty$. A/R contains the element $x = \sum_{i=1}^{\infty} i^{-1} u_i$ since $\sum i^{-2} = \pi^2/6$, but there is no element

Received by the editors November 27, 1950.

[1] Numbers in brackets refer to the references cited at the end of the paper.

$\sum i^{-1}e_i$ in A, for $\sum i^{-1}$ diverges. Therefore there is no subalgebra of A isomorphic to A/R.

It can be shown [1] that the radical of A is one-dimensional. Thus no restriction on the dimension of the radical will suffice. However it will now be shown that it is sufficient for A/R to be finite-dimensional.

THEOREM 1. *If A is a Banach algebra, R its radical, and A/R is finite-dimensional, then there is a subalgebra S of A isomorphic and homeomorphic to A/R. A is the vector space direct sum $S+R$.*

LEMMA 1. *If A is a Banach algebra, R its radical, and $\{u_i\}$ a denumerable set of pairwise orthogonal idempotents of A/R, then there exist idempotents e_i in A mapping on u_i via $A\to A/R$, and the e_i are pairwise orthogonal.*

The proof is by induction. Let a_1 be an element of A mapping on the class u_1. Then $a_1^2-a_1=r_1$ in R by hypothesis. For any r in R there exists $(1+4r)^{-1/2}=1-2r+6r^2-20r^3+\cdots$ since $\|r^n\|^{1/n}\to 0$ [2] guarantees convergence of this series. Define $e_1=(2a_1-1)[2(1+4r)^{1/2}]^{-1}+1/2=a_1(1-2r+6r^2-\cdots)+(r-3r^2+10r^3-\cdots)$. Then $e_1^2=e_1$ and e_1 maps on u_1 since a_1 does. Assume there exist $e_1,\cdots,e_{t-1}$ such that $e_i^2=e_i$, $e_ie_j=0=e_je_i$ for $i\neq j$, and $e_i\to u_i$, $i=1,2,\cdots,t-1$. Define $f=\sum_{i=1}^{t-1}e_i$. Then $f^2=f$, $e_if=fe_i$. Let b_t be any element such that $b_t\to u_t$. Define $a_t=(1-f)b_t(1-f)$. Then $e_ia_t=a_te_i=0$, and $a_t\to u_t$ since $fb_t\to 0$, $b_tf\to 0$, and $fb_tf\to 0$. Hence $a_t^2-a_t=r_t$ in R and $e_ir_t=r_te_i=0$, $i=1,2,\cdots,t-1$. Define $e_t=(2a_t-1)[2(1+4r_t)^{1/2}]^{-1}+1/2$. Then $e_t^2=e_t$, $e_t\to u_t$, and $e_te_i=e_ie_t=0$ since $e_ia_t=e_ir_t=0$. This completes Lemma 1.

LEMMA 2. *If A/R contains a ring direct sum $M_1\oplus M_2\oplus\cdots\oplus M_t$ of total matric algebras M_i, then A contains a ring direct sum of total matric algebras $S_i\to M_i$ via $A\to A/R$.*

Consider first a single matric algebra $M\subset A/R$, where M is generated over the complexes by u_{ij}, u_{ii} are pairwise orthogonal idempotents, $u_{ij}u_{jk}=u_{ik}$, and $u_{ij}u_{ks}=0$ for $k\neq j$. Since there are a finite number of u_{ii}, by Lemma 1 A contains idempotents $e_{ii}\to u_{ii}$ with $e_{ii}e_{jj}=e_{jj}e_{ii}=0$ for $i\neq j$. Choose an element $v_{i1}\to u_{i1}$ and an element $v_{1j}\to u_{1j}$. Since $u_{ii}u_{i1}u_{11}=u_{i1}$ and $u_{11}u_{1j}u_{jj}=u_{1j}$, v_{i1} may be chosen in $e_{ii}Ae_{11}$; v_{1j} may be chosen in $e_{11}Ae_{jj}$. Then $v_{1j}v_{j1}\to u_{1j}u_{j1}=u_{11}$. Hence $v_{1j}v_{j1}=e_{11}+a_j$ where a_j is in $R\cap e_{11}Ae_{11}$. By [3], a_j' exists. $(e_{11}+a_j')(e_{11}+a_j)=e_{11}+a_j'e_{11}+e_{11}a_j+a_j'a_j=e_{11}$ since $a_j'=\sum(-a_j)^n$ is also in $e_{11}Ae_{11}$. Define $e_{ij}=e_{i1}e_{1j}$. Then $e_{ij}e_{jk}=e_{ik}$ and $e_{ij}e_{hk}=0$ for $j\neq h$. Clearly e_{ij} is in A and $e_{ij} \to u_{ij}$. Thus A contains a total matric algebra (e_{ij}) isomorphic to M. The sum of the algebras S_i so constructed for each M_i is the ring direct sum since the basis elements are constructed from mutually orthogonal idempotents. This completes Lemma 2.

PROOF OF THEOREM. A/R is the direct sum of a finite number of finite-dimensional total matric algebras over the complex numbers. Hence A contains a subalgebra $S \cong A/R$. Since the isomorphism $S \to A/R$ is continuous, it is a homeomorphism. S is semi-simple; so $S \cap R = 0$. Therefore $S + R$ is a vector space direct sum.

When A/R is not finite-dimensional the theorem can still be proved if R is finite-dimensional and A/R is a well known type of algebra most generally defined in [4] as follows:

DEFINITION 2. The $B(\infty)$ direct sum of a denumerable number of algebras A_i is the completion in a specified norm of the algebra of all sequences $\{a_i\}$ such that a_i in A_i are 0 for all but a finite number of i.

THEOREM 2. *If A is a Banach algebra, the radical R of A is finite-dimensional, and A/R is the $B(\infty)$ direct sum of finite-dimensional total matric algebras, then A is a vector space direct sum, $A = B + C + D$, where B is finite-dimensional, $BC = CB = 0$, every idempotent of C mapping on an element in the basis of A/R is orthogonal to R, and $D \subset R$. When A is commutative, $D = 0$ and A is a ring direct sum of B and C.*

Let n be the dimension of R. Then there are at most n distinct primitive orthogonal idempotents e_k and n distinct primitive orthogonal idempotents e_s of A for which $e_k r_k \neq 0$ and $r_s e_s \neq 0$ for any r_k and r_s in R. Otherwise

$$e_{n+1} r_{n+1} = \sum_{k=1}^{n} \alpha_k e_k r_k, \qquad r_{n+1} e_{n+1} = \sum_{s=1}^{n} \beta_s r_s e_s$$

for complex α_k and β_s, since any $n+1$ elements of R are linearly dependent. However,

$$e_{n+1}(e_{n+1} r_{n+1}) = e_{n+1} r_{n+1} = \sum \alpha_k e_{n+1} e_k r_k = 0,$$
$$(r_{n+1} e_{n+1}) e_{n+1} = r_{n+1} e_{n+1} = \sum \beta_s r_s e_s e_{n+1} = 0.$$

Hence there are at most $2n$ primitive orthogonal idempotents e_j for which $e_j R \neq 0$ or $R e_j \neq 0$.

Let $\{u_{ij}\}$ be a basis for the matric algebras of A/R. Choose a fixed set of e_{ij} constructed as in Lemma 2 to map on u_{ij}, and number the set so that $e_j = e_{jj}$, $j = 1, \cdots, s$, are all idempotents of the set $\{e_{ij}\}$ which are not orthogonal to the radical. Define $e = \sum_{j=1}^{s} e_j$, $B = eAe$, $C = (1-e)A(1-e)$, and $D = eA(1-e) + (1-e)Ae$. Then $A = B + C + D$

is the usual two-sided Peirce decomposition of A. Obviously $BC = CB = 0$.

If A is commutative, $e(1-e)=0$; so $D=0$. Therefore A is a ring direct sum, $A = B \oplus C$.

Note that if $e_i = e_{ii}$ is an idempotent of $\{e_{ij}\}$ which is orthogonal to R and $e_k = e_{kk}$ is an idempotent of $\{e_{ij}\}$ which maps on $u_k = u_{kk}$ in the same matric algebra as u_{ii}, then e_{kk} is also orthogonal to R, since by Lemma 2 there exist e_{ik} and e_{ki} such that $e_{ki}e_{ii}e_{ik} = e_{kk}$. Then $e_{kk}R = e_{ki}e_{ii}e_{ik}R = 0$, and $Re_{kk} = Re_{ki}e_{ii}e_{ik} = 0$.

Let u be the image of e under $A \rightarrow A/R$. Then u is the sum $u = I_1 + \cdots + I_n$ where I_m is the unit element of a matric algebra in A/R. Now $D \rightarrow u(A/R)(1-u) + (1-u)(A/R)u$. Since u commutes with A/R, $D \rightarrow 0$. Therefore $D \subset R$. eAe/R is finite-dimensional and R is finite-dimensional. Therefore eAe is finite-dimensional. All idempotents of $\{e_{ij}\}$ not orthogonal to R are in B; so all idempotents of $\{e_{ij}\}$ in C are orthogonal to R. This completes Theorem 2.

The Principal Theorem of Wedderburn is known for finite-dimensional algebras, so $B = S_1 + R_1$. If it can be proved that $C = S_2 + R_2$, then it is proved for A; for $S = S_1 + S_2$ is a subalgebra, and it follows from $BC = CB = 0$ that $S_1 S_2 = S_2 S_1 = 0$, which implies $S_1 + S_2 \cong A/R$.

A C^*-algebra is a Banach algebra with a conjugate linear involution $x \rightarrow x^*$ such that $(xx^*)'$ exists for all x and $\|xx^*\| = \|x\|^2$. It is proved in [4] that a completely continuous C^*-algebra is the $B(\infty)$ direct sum of finite-dimensional total matric algebras.

THEOREM 3. *If A/R is a completely continuous C^*-algebra and R is finite-dimensional, then A is a vector space direct sum, $A = S + R$, of R and an algebra S isomorphic and homeomorphic to A/R.*

Theorem 2 applies to give $A = B + C + D$. The remark above implies a continuous isomorphism between S_1 and B/R_1. By the closed graph theorem this is a homeomorphism; so it remains to prove the theorem only for the algebra C in which every idempotent of the set $\{e_{ij}\}$ is orthogonal to R. It will thus be assumed that all idempotents in the set $\{e_{ij}\}$ are orthogonal to R.

LEMMA 3. *All elements of $\{e_{ij}\}$ are orthogonal to R.*

Since $e_{ij} = e_{ii}e_{ij} = e_{ij}e_{jj}$, and it has been assumed that all idempotents are orthogonal to R, it is clear that all e_{ij} are.

LEMMA 4. $\|e_{ij}\| = \|u_{ij}\| = 1$.

By [5, Theorem 10] and [4] the basis $\{u_{ij}\}$ may be chosen so that $u_{ij}^* = u_{ji}$.

$\|u_{ii}u_{ij}^*\| = \|u_{ii}^2\| = \|u_{ii}\| = \|u_{ii}\|^2$. Hence $\|u_{ii}\| = 1$.
$\|u_{ij}u_{ij}^*\| = \|u_{ij}\|^2 = \|u_{ij}u_{ji}\| = \|u_{ii}\| = 1$. Hence $\|u_{ij}\| = 1$. By definition,

$$\inf_{r \in R} \|e_{ii} + r\| = \|u_{ii}\| = 1.$$

Let n be the dimension of R. Then $r^{n+1} = 0$.

$$\begin{aligned} \|(e_{ii} + r)^{n+1}\| &= \|e_{ii}^{n+1} + (n + 1)e_{ii}^{n}r + \cdots + r^{n+1}\| \\ &= \|e_{ii}\| \leqq \|e_{ii} + r\|^{n+1}. \end{aligned}$$

For any $\epsilon > 0$ there is an r in R for which $\|e_{ii} + r\|^n < 1 + \epsilon$. Hence $\|e_{ii}\| \leqq 1$. Since $\|e_{ii}\| \leqq \|e_{ii}\|^2$, $\|e_{ii}\| = 1$. Now

$$\inf_{r \in R} \|e_{i1} + r\| = \|u_{i1}\| = 1,$$

$$\inf_{r \in R} \|e_{ij} + r\| = \|u_{ij}\| = 1,$$

$$e_{i1} = (e_{i1} + r)e_{11},$$

$$\|e_{i1}\| \leqq \|e_{i1} + r\| \|e_{11}\| = \|e_{i1} + r\|,$$

$$e_{1j} = e_{11}(e_{1j} + r),$$

$$\|e_{1j}\| \leqq \|e_{11}\| \|e_{1j} + r\| = \|e_{1j} + r\|.$$

This shows $\|e_{i1}\| \leqq 1$ and $\|e_{1j}\| \leqq 1$. The mapping $e_{ij} \rightarrow u_{ij}$ depresses the norm.

$$\|e_{ij}\| \leqq \|e_{i1}\| \|e_{1j}\| \leqq 1.$$

Therefore $\|e_{ij}\| = 1$.

Proof of theorem. A/R is the $B(\infty)$ direct sum of finite-dimensional total matric algebras M_i. By Lemma 4, A contains a subalgebra S_i equivalent to M_i. It will be shown that the map of any finite sum $\sum_{i=1}^{t} N_i$, N_i in S_i, into A/R is an isometry. Suppose $N_i \rightarrow \overline{N}_i$ in M_i, and I_i is the identity matrix of S_i. Since A/R is a C^*-algebra, $(\overline{I}_1\overline{I}_1^*)^* = \overline{I}_1 = \overline{I}_1\overline{I}_1^*$, $\|\overline{I}_1\| = \|\overline{I}_1\overline{I}_1^*\| = \|\overline{I}_1\|^2$; so $\|\overline{I}_1\| = 1$. Furthermore

$$\|\overline{I}_1 + \cdots + \overline{I}_t\| = \|[(\overline{I}_1 + \cdots + \overline{I}_t)(\overline{I}_1^* + \cdots + \overline{I}_t^*)]^*\|,$$

$$\|\overline{I}_1 + \cdots + \overline{I}_t\| = \|\overline{I}_1 + \cdots + \overline{I}_t\|^2 = 1.$$

Define $I = I_1 + \cdots + I_t$. Then

$$\inf_{r \in R} \|I + r\| = \|\overline{I}\| = 1.$$

$$\|(I + r)^{n+1}\| = \|I\| \leqq \|I + r\|^{n+1}.$$

Hence $\|I\| = 1$, and similarly

$$\inf_{r\in R}\left\|\sum_{i=1}^{t} N_i + r\right\| = \|\sum \overline{N}_i\|,$$

$$I(\sum N_i + r) = \sum N_i,$$

$$\|\sum N_i\| \leqq \|I\|\|\sum N_i + r\| = \|\sum N_i + r\|,$$

$$\|\sum N_i\| \leqq \|\sum \overline{N}\|.$$

Since the mapping $A \rightarrow A/R$ depresses norms,

$$\|\sum N_i\| = \|\sum \overline{N}_i\|.$$

This shows that the mapping of any finite sum $\sum_{i=1}^{t} N_i$ into A/R is an isometry. Let S be the $B(\infty)$ direct sum of the subalgebras S_i of A. Since A is complete, $S \subset A$. A dense subset of S maps isometrically and isomorphically onto a dense subset of A/R; therefore S is isomorphic and isometric to A/R. This proves Theorem 3.

The theorem will now be proved for an algebra in which the mapping $A \rightarrow A/R$ depresses the norm as little as possible.

Definition 3. An l_1 algebra is the commutative Banach algebra of all sums $\sum \alpha_i u_i$, where α_i are complex, u_i are a denumerable number of primitive orthogonal idempotents, and $\| \sum \alpha_i u_i\| = \sum |\alpha_i| < \infty$.

Theorem 4. *If A/R is an l_1 algebra and R is finite-dimensional, then $A = S + R$ where S is a subalgebra of A isomorphic and homeomorphic to A/R.*

As in Theorem 3 it is sufficient to consider an algebra A in which each idempotent e_i is orthogonal to R.

There exist pairwise orthogonal idempotents $e_i \rightarrow u_i$ by Lemma 1. The proof of Lemma 4 shows $\|e_i\| = 1$. For any $x = \sum \alpha_i e_i$ in A, $\|x\| \leqq \| \sum \alpha_i e_i\| \leqq \sum |\alpha_i| \|e_i\| = \sum |\alpha_i| = \| \sum \alpha_i u_i\|$. Now $x \rightarrow \sum \alpha_i u_i$, and the mapping $A \rightarrow A/R$ decreases norms. Hence $\|x\| = \sum |\alpha_i|$, that is, the mapping is an isometry on the completion S of the subalgebra of A generated by the e_i. Therefore S is isometric and isomorphic to A/R and $A = S + R$. This completes the proof.

In all the previous theorems the completion of the algebra generated by elements mapping on basis elements of A/R is disjoint from the radical. The following theorem shows this property is the essential one.

Theorem 5. *Suppose A is a Banach algebra, that the radical R is finite-dimensional, that A/R is the $B(\infty)$ sum of finite-dimensional total matric algebras, that S is the $B(\infty)$ sum in A of the matric algebras isomorphic to those of A/R, and that $S \cap R = 0$. Then S is isomorphic and homeomorphic to A/R, and A is the vector space direct sum $S + R$.*

S is complete and R is complete since the radical of a Banach algebra is closed. R is finite-dimensional so $S+R$ is complete. Also $(S+R)/R$ is complete; hence $A\rightarrow A/R$ maps $S+R$ onto A/R. $S\cap R=0$ implies $(S+R)/R=S$. Therefore $S\cong A/R$. The mapping $S\rightarrow A/R$ is 1-1 and continuous. By the closed graph theorem, S is homeomorphic to A/R. Suppose a in A maps on $[a]$ in A/R. Then there is an s in S which maps on $[a]$. Thus $a-s=r$ in R. Every $a=s+r$. Since S is semi-simple, $A=S+R$.

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South Bend, Ind.