NOTE ON THE LOCATION OF THE CRITICAL POINTS OF A REAL RATIONAL FUNCTION

J. L. WALSH

The object of this note is to study the critical points of a real rational function $p(z)$ of degree $-2m$ (>0) with $-2m$ poles in the closed interior of the unit circle C: $|z|=1$, with a real zero of multiplicity k (>0) in the point x_1 interior to C, and with a zero of multiplicity $-2m-k$ (>0) at infinity. The corresponding problems for the case $|x_1|>1$, and for the case that $p(z)$ is of degree k with infinity no longer a zero but a pole of $p(z)$ of multiplicity $k+2m$ (>0) with no restrictions on $|x_1|$, have already been treated elsewhere;[1] we retain the notation and terminology of that previous treatment.

We shall prove

LEMMA 5. *Let C: $|z|=1$ be the unit circle, let λ (>1) be constant, and let A: $z=x_1$ be a real point interior to C. If P is a variable nonreal point, we denote by Q the intersection other than P of the line AP with the circle through -1, $+1$, and P. Then the locus of points P: (x, y) such that we have $QP/AP=\lambda$ consists of the nonreal points of the circle*

$$(\lambda - 1)[(x - x_1)^2 + y^2] + x_1^2 - 1 = 0. \tag{9}$$

If P is the point (x, y), then the circle through -1, $+1$, and P is

$$X^2 + [Y - (x^2 + y^2 - 1)/2y]^2 = 1 + (x^2 + y^2 - 1)^2/4y^2, \tag{10}$$

where the running coordinates are X and Y. The point Q has the coordinates $(\lambda x_1+x-\lambda x,\ y-\lambda y)$, and a necessary and sufficient condition that Q lie on the circle (10) is precisely (9). Equation (9) represents a proper circle.

LEMMA 6. *Let C: $|z|=1$ be the unit circle, and A: $z=x_1$ a real point interior to C. If the point P: (x, y) lies exterior to the circle* (9), *then for every point Q' collinear with A and P, separated by A from P, and lying interior to the circle* (10), *we have $Q'P/AP<\lambda$.*

It is to be noted that A is the center of the circle (9). When P moves on AP monotonically away from A, the point Q of Lemma 5 moves monotonically toward A. Consequently the ratio QA/AP

Received by the editors October 23, 1950.

[1] J. L. Walsh, *The location of critical points*, Amer. Math. Soc. Colloquium Publications, vol. 34, §5.8.3. All references in the present note are to that book.

$=QP/AP-1$ decreases monotonically, and Lemma 6 follows.

We are now in a position to establish our main result:

THEOREM 8. *Let $p(z)$ be a rational function of degree $-2m$ all of whose poles lie in the closed interior of the unit circle C, which has a zero of multiplicity k (>0) in the real point x_1 interior to C, and which has a zero of multiplicity $-2m-k$ (>0) at infinity. Then all nonreal finite critical points of $p(z)$ exterior to C lie in the closed interior of the circle*

$$(k+2m)[(x-x_1)^2+y^2]+k(1-x_1^2)=0. \tag{11}$$

Let P be a nonreal critical point of $p(z)$ exterior to C, hence a position of equilibrium in the usual field of force. The particles at the poles of $p(z)$ total $2m$ in mass, are symmetric in the axis of reals, and lie in the closed interior of C; hence, by the method of proof of Lemma 2, the force these particles exert at P is equal to the force exerted at P by a single particle of mass $2m$ located at some point Q' of R (notation of Lemma 2). Then P is a position of equilibrium in the field due to this particle at Q', and to a particle of mass k in the point $A: z=x_1$. Consequently Q' lies in R_0 (notation of Lemma 2), P is collinear with A and Q', and we have $Q'P/AP=-2m/k$. It follows from Lemma 6 with $\lambda=-2m/k$ that P lies on or within the circle (11). Theorem 8 is established.

From a general result on circular regions (§4.2.4, Theorem 4), it follows that *under the hypothesis of Theorem* 8 *all real critical points of $p(z)$ lie in the interval S*:

$$(-2mx_1-k)/(-2m-k)\leqq z\leqq(-2mx_1+k)/(-2m-k).$$

In order to study the actual *locus* of critical points in Theorem 8, we consider a somewhat more general situation:

THEOREM 9. *Let a region R symmetric in the axis of reals be the locus of q (>4) poles of a real rational function $p(z)$. Then each point of R belongs to the locus of critical points of $p(z)$.*

As in the proof of §4.2.2, Theorem 1, we find it convenient to consider $R(z)\equiv 1/p(z)$, and shall prove that each (interior) point of R can be a critical point not a multiple zero of $R(z)$. This conclusion is valid for a real point α_0 of R, for let α_0 lie interior to R on the axis of reals; precisely the method of §4.2.2 then shows that α_0 can be a critical point not a multiple zero of $R(z)$.

Let now α_0 be a nonreal point of R; we set

$$R(z)\equiv(z-\alpha_0)(z-\bar{\alpha}_0)(z-\alpha)(z-\bar{\alpha})R_1(z),$$

where α is allowed to vary in the neighborhood of the fixed point α_0, but where the zeros and poles of $R_1(z)$ are fixed and remote from α_0. The critical points of $R(z)$ are the zeros of $R'(z)\equiv(z-\bar{\alpha}_0)(z-\alpha)\cdot(z-\bar{\alpha})R_1(z)+(z-\alpha_0)(z-\alpha)(z-\bar{\alpha})R_1(z)+(z-\alpha_0)(z-\bar{\alpha}_0)(z-\bar{\alpha})R_1(z)+(z-\alpha_0)(z-\bar{\alpha}_0)(z-\alpha)R_1(z)+(z-\alpha_0)(z-\bar{\alpha}_0)(z-\alpha)(z-\bar{\alpha})R_1'(z)=0$. This equation defines z as an implicit function of α, and the equation is satisfied for $z=\alpha=\alpha_0$. Even though z is not an analytic function of α, we have by differentiation for the values $z=\alpha=\alpha_0$

$$\frac{\partial R'(z)}{\partial z} = 2(\alpha_0 - \bar{\alpha}_0)^2 R_1(\alpha_0),$$

$$\frac{\partial R'(z)}{\partial \alpha} = -(\alpha_0 - \bar{\alpha}_0)^2 R_1(\alpha_0).$$

We interpret $R'(z)=0$ as two real equations in four real variables, the real and pure imaginary parts of z and α; the jacobian for the values $z=\alpha=\alpha_0$ has a value which is different from zero. It follows from the implicit function theorem for real variables that the equation $R'(z)=0$ defines z as a function of α, and when α varies throughout a neighborhood of α_0 then z also varies throughout a neighborhood of α_0; it is readily shown also from the proof of the implicit function theorem based on successive approximations that if α_0 is now allowed to vary over a small neighborhood, then a neighborhood of α_0 *of fixed size* as a locus of α corresponds to a neighborhood of α_0 as a locus of z which contains *uniformly a circle of constant positive radius* whose center is the variable α_0. The proof of Theorem 9 can now be completed as was the proof of §4.2.2, Theorem 1.

We return now to Theorem 8. If we replace the problem represented by Theorem 8 by the more general problem of finding the critical points of a function of the form $(0<k<-2m)$

$$p(z) \equiv (z-x_1)^k/(z-\alpha_1)^{m_1}(z-\bar{\alpha}_1)^{m_1}(z-\alpha_2)^{m_2}(z-\bar{\alpha}_2)^{m_2} \cdots (z-\alpha_n)^{m_n}(z-\bar{\alpha}_n)^{m_n}, \tag{12}$$

where we have $|x_1|<1$, $|\alpha_j|\leqq 1$, $m_j>0$, $m_1+m_2+\cdots+m_n=-m$, and where k, m, x_1 are given but k, m_1, m_2, $\cdots$, m_n, m need no longer be integral, then *the interior of* C *plus the closed interior of* (11) *plus* S *plus the point at infinity is the precise locus of critical points of* $p(z)$. Any point z interior to C can be a critical point of $p(z)$, as follows from Theorem 9. Any point of S can be a critical point of $p(z)$; compare §4.2.4, Theorem 4. Any nonreal point P in the closed interior of (9) not interior to C can be a critical point of $p(z)$, for if P is given there exists a point Q' on the line PA extended, which is contained in the closed region R_0 (notation of Lemma 2) and with $Q'P/AP = -2m/k$; it follows from the proof of Lemma 2 that a suitable choice of negative particles in C of total mass $2m$ is equivalent (so far as concerns the force at P) to a $(2m)$-fold negative particle at Q'; thus P is a critical point of a suitably chosen $p(z)$ of type (12). Every real point interior to (11) lies in C or S, and the conclusion of Theorem 8 persists, so the locus of critical points of (12) is as stated.

Under the original conditions of Theorem 8, with x_1 given, every real point interior to C belongs to the locus of critical points if we have $-m=1$, and every point interior to C belongs to the locus if we have $-m>1$.

If the hypothesis of Theorem 8 is modified slightly, the conclusion requires large modification:

THEOREM 10. *Let $p(z)$ be a rational function of form* (12) *where we have $|x_1|<1$, $|\alpha_j|\leqq 1$, $m_j>0$, $m_1+m_2+\cdots+m_n=-m$, and where m, x_1 are given, but k, m_1, m_2, $\cdots$, m_n, m need not be integral. Then the locus of critical points of $p(z)$ for all possible choices of k, m_1, m_2, $\cdots$, m_n, α_j, consists of the extended plane.*

The proof of Theorem 10 is similar to the previous proof of the locus property of (12), and is left to the reader.

HARVARD UNIVERSITY