TWO-DIMENSIONAL SUBGROUPS[1]

DEANE MONTGOMERY AND LEO ZIPPIN

1. **Introduction.** The authors have recently shown (see [1], bibliography) that every separable metric, connected, locally compact, noncompact, n-dimensional group G with $n \geqq 1$ has a subgroup isomorphic to the real numbers. The subgroup is understood to be closed in the topological as well as in the algebraic sense. In the present paper it will be shown that if $n \geqq 2$, the group G has a noncompact, connected, two-dimensional subgroup.

The analogous result is not true for compact groups; witness the proper rotation group 0_3 of three-space and its universal covering group. However, these are the only exceptions. It is a corollary of the result of this paper and known facts about compact groups [7; 8] that 0_3 and its covering group are the only connected finite-dimensional locally compact groups of dimension greater than two which fail to contain two-dimensional subgroups.

We shall use the main result and follow the notation of the earlier paper [1]. The brief summary of supporting ideas which is given in §2 of that paper may be helpful in the reading of this one and references to [1] are often to this summary. However, this paper can be followed without acquaintance with the details of the other.

An outline of the proof follows. In order to prove our theorem it is sufficient to prove (Theorem 2) that every noncompact connected n-dimensional group, $n > 2$, has a noncompact connected subgroup of dimension greater than one and less than n. For this it is sufficient to consider the case where G has no center and no invariant one-dimensional subgroup. We assume Theorem 2 false and consider the n-dimensional connected locally connected group L which is mapped into an everywhere dense subgroup of G by a one-one continuous homomorphism. The group L has a subgroup Q isomorphic to the reals.

We come now to the principal device of the proof. The assumption that Theorem 2 is false implies that the set of conjugates of Q^+, one of the semi-groups of Q, fills out L with no overlap except at e. These transforms have a cross section S with at least certain homotopy properties of an $(n-1)$-sphere. Hence S can have no nontrivial covering because $n-1$ is at least two. But on the other

Presented to the Society, November 25, 1950; received by the editors October 12, 1950.

[1] This work was done while one of the authors was under contract with ONR.

hand it is shown that S does have a covering. In this way the assumption that the theorem is false leads to a contradiction.

2. The principal theorem.

THEOREM 1. *Let G be a connected, separable metric, locally compact, noncompact n-dimensional group, $n>1$. Then G contains a two-dimensional connected noncompact subgroup.*

The principal theorem may be derived from the following by a simple induction on dimension.

THEOREM 2. *Let G be a connected, separable metric, locally compact, noncompact n-dimensional group, $n>2$. Then G contains a connected noncompact closed subgroup H with*

$$1 < \dim H < n.$$

We shall prove Theorem 2 by a reductio ad absurdum. To this end we formulate the following assumption, whose contradiction establishes Theorem 2. It is always assumed that G is separable metric and locally compact.

ASSUMPTION (A). *There exists a noncompact n-dimensional group G, $n>2$, containing no connected subgroup H which is not compact and for which*

$$1 < \dim H < n.$$

We shall show first that (A) is equivalent to a formally stronger proposition in which the group G is assumed to have only a trivial center. We show this in a sequence of lemmas based on (A).

3. Consequences of Assumption (A).

LEMMA 3.1. *G contains no connected one-dimensional normal subgroup.*

Suppose N to be a connected one-dimensional normal subgroup of G. The group N is abelian [2] and G/N is finite-dimensional [3]. There are two possible cases. First, if N is compact, we know that G/N is *not* compact, and in this case G/N contains a subgroup isomorphic to the reals [1]. Let this group be denoted by F. In the second case N is not compact and N is itself isomorphic to the reals. Now, in this second case, G/N is at least one-dimensional and contains some connected one-dimensional subgroup; let this subgroup be denoted by F. In either case let H denote the subgroup $f^{-1}(F)$ where f is the natural map

$$f: G \rightarrow G/N.$$

It is clear that in each case the group H is connected and not compact. Moreover, it is at most two-dimensional since F and N are both one-dimensional [1; 4]. Since in each case N is a proper subgroup of H, it follows that H is at least two-dimensional. Then the existence of N leads to a contradiction of (A) since H is a two-dimensional noncompact group.

COROLLARY 3.11. *The center of G is zero-dimensional.*

An abelian group of positive finite dimension contains one-dimensional subgroups [8] and every subgroup of the center is normal. This together with the preceding lemma proves the corollary.

Let Z be the center of G.

LEMMA 3.2. *The group G/Z has no center but the identity.*

By 3.11, dim $Z=0$. Let f be the map

$$f: G \to G/Z.$$

The group G/Z is finite-dimensional and its center will be denoted by Z^*. Let N be the identity component of $f^{-1}(Z^*)$,

$$N = C_e f^{-1}(Z^*),$$

so that N is a connected normal subgroup of G. By 3.1, N is not one-dimensional. Assume for the moment that

$$1 < \dim N < \dim G$$

so that by (A), N is compact. Now $N \cap Z$ is zero-dimensional and

$$N/(N \cap Z) \subset Z^*$$

is abelian, so by known theorems on compact groups [7; 8] N is abelian, and is therefore central [6]. This is impossible by 3.11. It follows that dim $N=0$ or n.

If dim $N=0$, then $N=e$ and $f^{-1}(Z^*)$ is zero-dimensional. But $f^{-1}(Z^*)$ is invariant and therefore central. Hence $f^{-1}(Z^*) \subset Z$, and since $f(Z)$ is the identity of G/Z, this is the assertion of the lemma.

The only case left to consider is where

$$\dim N = \dim G = n > 2.$$

In this case $N=G$ [1]. This means that G/Z is abelian. Since Z is zero-dimensional, G/Z is at least three-dimensional. We distinguish two subcases. First if G/Z is not compact, then by the known structure of abelian groups [8] it contains a proper noncompact connected subgroup H of dimension more than one. Then the identity

component of $f^{-1}(H)$ is a noncompact two- (or more) dimensional proper subgroup of G and this contradicts Assumption (A). There remains the case where G/Z is compact as well as abelian.

In this case let H be a compact connected two-dimensional subgroup of G/Z and let $N = C_e f^{-1}(H)$. Now $1 < \dim N < G$, so N is compact. Then since N is also normal and abelian, it is necessarily central. This contradicts 3.11 and proves the lemma.

LEMMA 3.3. *The group G/Z contains no noncompact connected subgroup H^* such that*

$$1 < \dim H^* < \dim G/Z.$$

If such a subgroup exists, it is a proper subgroup of the finite-dimensional group G/Z and

$$H = f^{-1}(H^*)$$

is a noncompact proper subgroup of G. Since Z is zero-dimensional, H is of at least as large a dimension as H^* so that

$$1 < \dim H < \dim G.$$

Now

$$\dim C_e(H) = \dim H$$

so that $C_e(H)$ must be compact by (A). Thus the noncompact H must contain a compact open subgroup H_1, and

$$f(H_1)$$

is a closed, in fact compact, subgroup of H^*. Therefore, since H^* is connected,

$$\dim f(H_1) < \dim H^*.$$

Since $f(H)$ is the union of countably many sets homeomorphic to $f(H_1)$, $\dim f(H) = \dim f(H_1)$. However this is a contradiction, since $f(H) = H^*$ and this proves the lemma.

LEMMA 3.4. *The group G/Z is not compact.*

It is shown in the proof of 3.2 that G/Z cannot be compact abelian, and we know that $\dim G/Z \geqq 3$.

Let T be a subgroup of G which is isomorphic to the real numbers, and let

$$F = f(T)$$

so that $\overline{F}$ is a connected closed subgroup of G/Z. Suppose that G/Z

is compact. If dim G/Z exceeds 3, then G/Z is the limit of a sequence of Lie groups G_i where dim G_i=dim $G/Z>3$. It follows from the known classification of compact simple Lie groups [8] that rank G_i =rank $G\geqq 2$ (where rank is the maximum dimension of abelian subgroups). This can be used to show that $\overline{F}$ is in an abelian proper subgroup H of G/Z where

$$\dim H \geqq 2.$$

As before, consideration of $f^{-1}(H)$ leads to a contradiction.

Now dim $G/Z>2$ since dim $G>2$ and dim $Z=0$. Thus there remains to investigate only the case that dim $G/Z=3$, G/Z compact. If G/Z has rank 2 or more, we obtain a contradiction as above.

If G/Z is of rank one, then it is one of the two well known simple Lie groups, the rotation-group of three-space or its three-sphere universal covering. Let Z^* be a compact open subgroup of Z. Then Z^* is invariant and the group G/Z^* is not compact. There is a natural map from G/Z^* to G/Z which is an infinite covering map, but there is no such covering when G/Z is either of the groups under consideration. Hence in either case G could not have been connected and noncompact and this final contradiction completes the proof of the lemma.

It follows from the preceding lemmas that Assumption (A) implies the following formally stronger assumption.

ASSUMPTION (B). *There exists a connected group G which is not compact, is of dimension $n>2$, has only a trivial center and no normal one-dimensional subgroup, and contains no connected subgroup H which is not compact for which*

$$1 < \dim H < n.$$

In the sequel we shall operate on Assumption (B). The contradiction of (B) ultimately gives a proof of our principal theorem.

If x is in G, the symbol G_x denotes the set of those elements of G which commute with x and

$$K_x = C_e(G_x).$$

The subgroups G_x and K_x are closed. We also use the symbol $g[x]$ as follows:

$$g[x] = gxg^{-1}.$$

4. Consequences of Assumption (B).

LEMMA 4.1. *Let T_1 be isomorphic to the reals and T_2 be a distinct connected abelian group, or abelian local group. Then*

$$T_1 \cap T_2 = e.$$

This is clear enough, because if T_1 and T_2 contained a point z, the group G_z would be at least two-dimensional and noncompact. Since there is no nontrivial center in G, either $z=e$ as is to be proved *or* G_z is of dimension less than n, and we have contradicted (B). This completes the proof.

LEMMA 4.2. *Let K be a connected noncompact subgroup of G and let G_K be the normalisor of K. Then K is the identity component of G_K,*

$$K = C_e(G_K).$$

In the light of Assumption (B) the only case to consider is $\dim K = 1$. If

$$\dim G_K > \dim K = 1,$$

then $\dim G_K < n$ contradicts (B) and

$$\dim G_K = n$$

implies that K is invariant, and this is impossible by (B). This completes the proof.

LEMMA 4.3. *If T, in G, is isomorphic to the reals, then for any $x \in T$, $x \neq e$,*

$$T = K_x.$$

By (B), x is not central, and so $\dim G_x < n$. Since $T \subset K_x \subset G_x$, it follows by (B) again that

$$\dim G_x \leqq 1$$

and therefore

$$\dim G_x = 1.$$

It is now clear that

$$T = K_x$$

and the proof of the lemma is complete.

For the next lemmas, let T denote a fixed subgroup of G isomorphic to the reals and let

$$S = G[T]$$

be the totality of elements belonging to transforms gTg^{-1} of T, $g \in G$.

LEMMA 4.4. *Let $x \in \bar{S}$, $x \neq e$, where $\bar{S}$ is the closure of $S = G[T]$. Then*

dim $G_x=1$, *K_x is isomorphic to the reals, and* $K_x \subset \overline{S}$.

Since $x \neq e$ and $x \in \overline{S}$, there exists a sequence of elements

$$x_n \in S, \qquad x_n \neq e$$

which converges to x. For each x_n the associated G_{x_n} contains a transform of T and is at least one-dimensional. By the preceding lemma it is then exactly one-dimensional and each identity-component K_{x_n} is in fact a suitable transform $g_n T g_n^{-1}$ of T, for some $g_n \in G$. It is clear that a subsequence of these groups must converge to some subgroup of G_x with a component extending from e to infinity. (Note that it is not asserted that the limit is connected nor that x belongs to K_x.) Accordingly K_x is not compact and dim $K_x > 0$. From this it follows as above that

$$\dim K_x = 1,$$

and then K_x is isomorphic to the reals since it is not compact. Clearly every point of K_x is a limit of points of S, and $K_x \subset \overline{S}$. This proves the lemma.

It may conceivably happen that x is in $\overline{S}$ but not in S and even that K_x is in S, although G_x is only in $\overline{S}$.

Now let L denote the n-dimensional connected, locally compact, locally connected group associated [1] with G, and let α be the one-one continuous homomorphism taking L into G,

$$\alpha\colon L \to G.$$

If G is locally connected, then L coincides with G and α is the identity map. In general, L is a distinct group and the image $\alpha(L)$ is everywhere dense in G. This set $\alpha(L)$ is algebraically closed, and is uniquely determined as the maximal arc-wise connected subset of G which contains e.

The group T and every gTg^{-1}, g in G, is contained in $\alpha(L)$. Thus

$$S = G[T] \subset \alpha(L).$$

Now let

$$Q = \alpha^{-1}(T)$$

and let

$$P = L[Q].$$

The set Q is a subgroup of L isomorphic to the reals and P is a subset of L such that

$$\alpha(P) \subset S.$$

This last is because every group

$$mQm^{-1} \subset L, \qquad m \text{ in } L,$$

is mapped by α into

$$gTg^{-1} \subset S$$

where $g=\alpha(m)$.

Let T^+ denote a fixed one of the two semi-groups of T which we may think of, say, as corresponding to the non-negative reals.

Let

$$Q^+ = \alpha^{-1}(T^+)$$

and let

$$P^+ = L[Q^+].$$

It can be seen as above that

$$\alpha(P^+) \subset S^+ = G[T^+].$$

In the next section we shall show that P^+ fills out L. Here we establish some preliminary lemmas.

LEMMA 4.5. *Let y in L, $y \neq e$, belong to the closure of P^+. Then K_y, the identity component of L_y, belongs to the closure of P, and K_y is isomorphic to the reals.*

Choose elements y_n in P^+, $y_n \neq e$, such that y_n approaches y. Take g_n in L and q_n in Q^+ such that

$$y_n = g_n q_n g_n^{-1}$$

and define the noncompact connected set Q_n by

$$Q_n = g_n Q g_n^{-1}.$$

The sets Q_n have a subsequence converging sequentially to an abelian group which contains y, and this group will be denoted by H for the moment. Then H belongs to the closure of P and H has a component K which is not compact. Thus dim $K \geqq 1$. On the other hand

$$K \subset K_y$$

and

$$\alpha(K_y) \subset G_x$$

for that element x in G for which

$$x = \alpha(y).$$

Since $\alpha(y)$ is continuous, it follows that x belongs to the closure of $\alpha(P)$. Since S contains $\alpha(P)$, we know from a preceding lemma that

$$\dim G_x = 1$$

and it follows that

$$1 \leqq \dim K \leqq \dim K_y \leqq 1$$

and that

$$K_y = K$$

is isomorphic to the reals. This completes the proof.

COROLLARY. *If b in L, $b \neq e$, belongs to the closure of P^+, then the group generated by b and K_b is abelian.*

This follows because the group considered in the corollary is contained in the abelian group H used in the proof of the lemma.

The element b may or may not be in $K = K_b$. The symbol bK denotes K if b is in K; otherwise bK denotes a coset of K.

LEMMA 4.6. *Let b be an element of L, $b \neq e$, with b in the closure of P^+. Then $K = K_b$ is isomorphic to the reals and belongs to the closure of P. For any g in L the intersection*

$$bK \cap g(bK)g^{-1}$$

is vacuous, or is the identity element, or it is the coset bK.

For the most part this is a restatement of the preceding lemma and corollary. For the proof of the final statement observe that any

$$z \in bK \cap g(bK)g^{-1}$$

commutes with every element of bK and with every element of $g(bK)g^{-1}$. If L_z is one-dimensional, it cannot contain two intersecting lines. Hence if bK and $g(bK)g^{-1}$ are distinct and contain z, then L_z is at least two-dimensional and noncompact. If z is not e, let

$$y = \alpha(z), \qquad y \neq e.$$

Then K_y is at least two-dimensional and noncompact. This contradicts Assumption (B) and completes the proof.

5. **The set** $L[bK]$. Continuing the notation of the preceding

lemma, we shall investigate the set of elements lying on transforms of the coset bK. It is to be understood as remarked above that if b is in K then $bK=K$. In this way we can discuss the general case without distinguishing two possible situations.

Let L_K denote the normalisor of K in L. Then $\alpha(L_K)$ is contained in the normalisor of $\alpha(K)$ in G and therefore, in view of (B), $\alpha(K)$ is the identity component of $\alpha(L_K)$. Therefore K is the identity component of L_K since α is one-one.

It follows that L_K/K is totally disconnected. Then there exists a subgroup K' of L_K which is open and closed in L_K and such that

$$K'/K$$

is compact. We can regard K'/K as a compact group of automorphisms of K. Since K is isomorphic to the reals, there must be an open and closed subgroup K^* of L_K such that every element of K^* commutes with every element of K.

Let K^{**} denote the subgroup of K^* which leaves bK invariant. We can regard K^{**}/K as a compact transformation group of bK. Since bK is topologically a line, the transformation group K^{**}/K is effectively a finite group of order two at most. This means that there is an open subgroup K^{***} of K^{**} such that every element of K^{***} commutes with every element of bK.

This proves the following lemma.

LEMMA 5.1. *There exists a compact symmetric $U(e)$ in L such that if g is in U^2 and leaves K invariant, then it commutes with every element of K, and if furthermore it leaves bK invariant, then it commutes also with every element of bK.*

In the sequel we shall suppose that U is a neighborhood of the identity as described in the preceding lemma, and we shall consider the sets

$$U[b] \quad \text{and} \quad U[bK].$$

Suppose that a_i is in U, $i=1, 2$, that b_i is in bK, $i=1, 2$, and that

$$a_1b_1a_1^{-1} = a_2b_2a_2^{-1}.$$

Then

$$a_2^{-1}a_1b_1a_1^{-1}a_2 = b_2$$

and

$$b_2 \in bK \cap abKa^{-1}$$

where a is in U^2, $a=a_2^{-1}a_1$.

It follows from an earlier lemma that $abKa^{-1}$ coincides with bK. The set $abKa^{-1}$ is the same as the set $aba^{-1}\cdot aKa^{-1}$. Let

$$b' = aba^{-1}.$$

Since $b\in bK$, $b'=aba^{-1}\in abKa^{-1}=bK$. Then b' is in bK and b' commutes with every element of K by Lemma 4.2. Also b' commutes with every element aKa^{-1} since for $k\in K$

$$b'aka^{-1} = aba^{-1}aka^{-1} = abka^{-1} = akba^{-1} = aka^{-1}aba^{-1} = aka^{-1}b'.$$

This implies that

$$aKa^{-1} \quad \text{and} \quad K$$

must coincide, for otherwise $L_{b'}$ would be of dimension at least 2 and noncompact. Then

$$a \in L_K$$

and, since $a\in U^2$, it follows from the preceding lemma that a commutes with every element of K.

LEMMA 5.2. *If $a_i\in U$, $b_i\in bK$, $i=1, 2$, $b_1\neq e$, and*

$$a_1b_1a_1^{-1} = a_2b_2a_2^{-1},$$

then

$$b_1 = b_2.$$

Furthermore

$$a_1bka_1^{-1} = a_2bka_2^{-1}$$

for every k in K.

It was shown above that the element $a=a_2^{-1}a_1$ commutes with every element of K and also that a leaves bK invariant. Therefore it belongs to K^{**} and so by the choice of U according to Lemma 5.1 it follows that a commutes with every element of K and of bK. This proves the lemma.

LEMMA 5.3. *The set $\overline{U}[b]$ is at least $(n-1)$-dimensional and the set $\overline{U}[bK]$ is n-dimensional.*

The set $\overline{U}[b]$ is a continuous image of the compact n-dimensional set $\overline{U}$ and inverse images have dimension at most one. This proves the first part of the lemma.

Let N be a closed interval in K including e and with the property that

$$e \text{ is not in } bN.$$

This is possible since $b \neq e$. The set $\overline{U}[bN]$ has the structure of a topological product of $\overline{U}[b]$ and N, as we shall now see.

Let a_1, a_2 be in U and n_1, n_2 be in N which is in K, and assume

$$a_1 b n_1 a_1^{-1} = a_2 b n_2 a_2^{-1}.$$

Note that bn_1 cannot be e by the choice of N. Then by Lemma 5.2

$$bn_1 = bn_2, \qquad n_1 = n_2,$$

and also

$$a_1 b n a_1^{-1} = a_2 b n a_2^{-1}$$

for every n in N (in fact for every n in K).

Thus, if $a_1 b a_1^{-1}$ and n are given, the point $a_1 b n a_1^{-1}$ is uniquely determined and in this way there is given a map from the set of pairs $(a_1 b a_1^{-1}, n)$ to the set $\overline{U}[bN]$. The relations above show also that the map is one-one.

LEMMA 5.4. *It is a consequence of* (B) *that the set* $P^+ = L[Q^+]$ *is all of* L.

It follows from the preceding result that $L[Q^+]$ is n-dimensional. Since L is locally connected, it follows that $L[Q^+]$ has inner points and if it were not all of L, it would have boundary points which belonged to an at most $(n-1)$-dimensional invariant set. But if b denotes such a boundary point, the preceding lemmas have shown that there is also some coset bK in the boundary and $L[bK]$ is n-dimensional. This is not possible and the lemma is proved.

6. **The group** L_Q. We have shown above that the group L is completely covered by transforms gQ^+g^{-1} with g in L and Q^+ a semi-group of a fixed group Q isomorphic to the reals. Two distinct semi-groups, or rays as we shall sometimes call them, have only the identity in common.

Observe for later use that the fact that $L[Q^+]$ fills out L implies the existence of an element g in L such that

$$gQ^+g^{-1} = Q^-.$$

This means, of course, that the normalisor L_Q of Q is larger than Q, although as we know,

$$Q = C_e(L_Q).$$

The fact that $L[Q^+]$ fills out L implies also that all elements of L, except e, generate groups isomorphic to the integers. Hence:

LEMMA 6.1. *The group L contains no elements of finite order and no nontrivial compact subgroups.*

LEMMA 6.2. *The group L_Q/Q is discrete.*

The fact that G has no center but e implies the same for L. Since every element of L lies on a reals group and every x has a one-dimensional G_x, it follows that no element of L not on Q can commute with every element of Q. The group L_Q has Q as identity component and it follows, as in the argument preceding Lemma 5.1, that L_Q contains an open subgroup made up of elements commuting with all of Q. This subgroup must be Q and hence Q is open in L_Q as was to be proved.

LEMMA 6.3. *Let $q \neq e$ be in Q. Then $L[q]$ is a local section of the rays (in the sense of and as proved in* 5.2) *and $L[q]$ meets Q^+, and every ray, in an infinity of points.*

Take g in L so that

$$gQ^+g^{-1} = Q^-.$$

Then

$$g^2Q^+g^{-2} = Q^+.$$

However g^2 cannot commute with all of Q^+ for if it did g^2 would be in Q as well as in a reals group through g; so g^2 must be a magnification on Q^+ and successive iterates carry a point x of Q toward infinity or e. This proves the lemma.

7. **The cross section** S. The family of curves $L[Q-e]$ fills out $L-e$. Each element of L has a unique root and power of any order and this enables us to define for each $r>0$ and each x in L an element

$$x^r$$

in L which depends continuously on x and r. In this way R, the multiplicative group of positive real numbers, operates as a transformation group on $L-e$. The operation on $L-e$ will be shown in a moment to have a property which we have defined earlier [5] as dispersive. The action of r on x is denoted by $r\{x\}$, that is

$$r\{x\} = x^r.$$

In the present context the property of being dispersive means: points x and y in $L-e$ are interior to compact sets A and B in $L-e$ and there are positive real numbers r_0 and r_1 such that if $r>r_0$ or $r<r_1$, then

$$B \cap r\{A\} = 0.$$

It is easy for integral r and can be shown for all $r>0$ that for any elements g and x, $gx^rg^{-1}=(gxg^{-1})^r$, that is,

$$g[r\{x\}] = r\{g[x]\}.$$

The dispersive property will now be verified. If x is in $Q-e$, then from 5.3, $U(e)$ may be so chosen that $\overline{U}[x]$ is a cross section of $\overline{U}[Q-e]$. By the remark above,

$$U[r\{x\}] = r\{U[x]\}.$$

Let T be a compact interval of R which includes $r=1$ as an interior point. Then $\overline{U}[T\{x\}]=A$ is a compact set containing x in its interior, and

$$r\{A\} = \overline{U}[(rT)\{x\}].$$

The set $r\{A\}$ is made up of points gx^rg^{-1}, g in $\overline{U}$, t in T. For any preassigned compact B in $L-e$, such points cannot touch B if r is sufficiently large or small.

Since R acts dispersively there is a cross section S of the orbits in the large [5]. Then S is a cross section of the semi-groups of $L-e$. If $U(e)$ is given in L with $\overline{U}$ compact, then B, the boundary of U, is a compact set touching each ray of $L-e$. The set S is the continuous image of B and hence S is compact. Topologically

$$L - e = S \times R.$$

We see now that any compact set in L can be shrunk to e. Hence L is simply connected, locally shrinkable, and all compact cycles in L bound in L.

Let g and h be any two distinct points in L. Then there is a unique group P isomorphic to the reals containing $e=g^{-1}g$ and $g^{-1}h$. In fact this is a group which is conjugate to Q. The coset gP contains g and h. This proves the following lemma.

LEMMA 7.1. *Let g and h be distinct points of L. Then there is a unique subgroup P of L which is isomorphic to the reals, one of whose cosets contains both g and h. The unique arc of this coset joining g and h may be parameterized by* (0, 1) *in a continuous way by making use of group parameters on P.*

The last part of the lemma about parameters merely notes the fact that if $e=g^{-1}g$ is to be zero and $g^{-1}h$ is to be one, then the group parametrization for the arc of P between these points is determined in a unique way.

Hence if g and h and p, $0 \leqq p \leqq 1$, are given, there is determined a unique point, say

$$\theta(g, h, p),$$

of the arc of the coset joining g and h, and θ is a continuous function of g, h, and p simultaneously with

$$\theta(g, h, 0) = g, \qquad \theta(g, h, 1) = h.$$

Let $\phi(t)$ and $\psi(t)$ be two continuous functions on $(0, 1)$ with values in L and let

$$\phi(0) = \psi(0), \qquad \phi(1) = \psi(1).$$

Then we may deform $\phi(t)$ to $\psi(t)$ in L while keeping $\phi(0)$ and $\phi(1)$ fixed. This can be done by moving each point $\phi(t)$ along the arc joining $\phi(t)$ and $\psi(t)$ with the help of θ as defined above.

Now let $\phi(t)$ be as above and let $\epsilon > 0$ be given. Then there is a $\psi(t)$ with the same end points as $\phi(t)$ such that $\psi(t)$ traces out a one-dimensional locus and such that ϕ may be deformed to ψ by an ϵ deformation which leaves end points fixed. To see this choose a finite set of points

$$t_0 = 0, \ t_1, \cdots, t_k = 1.$$

Let

$$\psi(t), \qquad t_i \leqq t \leqq t_{i+1},$$

trace out an arc joining $\phi(t_i)$ and $\phi(t_{i+1})$. Such an arc exists since L is connected and locally connected. The t_i's may be chosen so that the curve $\psi(t)$ just defined is in an ϵ neighborhood of $\phi(t)$ and so that $\phi(t)$ stays in such a neighborhood during the deformation as described above.

LEMMA 7.2. *For $n > 2$, S is simply connected.*

The fact that S is locally shrinkable follows from the same fact for L. The present lemma, however, needs further discussion.

Let $\phi(t)$ to be a closed path lying in S. The arcs used in defining $\psi(t)$ as discussed above may be chosen in S so that $\psi(t)$ is in S. This is because $\phi(t)$ is in S and because S is connected and locally connected. Then $\phi(t)$ is homotopic through an ϵ neighborhood of S

(in $L-e$) to a $\psi(t)$ in S with a one-dimensional image. This deformation can be thrown into S since $S\times R=L-e$. Hence we shall assume that the original $\phi(t)$ has a one-dimensional image.

The image of $\phi(t)$ cannot cover S whose dimension is $n-1>1$ and there is a point y in S such that y is not in $\phi(t)$. Let P be the unique reals group through e and y. Let a be the point of S such that $a=\phi(0)=\phi(1)$.

Now let C be a cross section of the cosets of P such that C satisfies

i. $L=C\times P$,

ii. $a\in C$,

iii. C crosses P on the opposite side of e from y.

Such a cross section exists [5] because P is a dispersive transformation group on L under right translations. The dispersive character of P is easily verified by the use of a right-invariant metric [9].

Then $\phi(t)$ may be deformed to a $\psi(t)$ in C leaving a fixed and without crossing e. This can be done by projecting on C along the cosets of P. The closed path $\psi(t)$ in C can be shrunk in C. This is because it can be shrunk in L keeping a fixed and this shrinking can be projected along cosets of P.

The set C does not contain e and hence we see that $\phi(t)$ is homotopic to a constant in the space $L-e$. This homotopy can be deformed to S since $L-e=S\times R$. This proves the lemma.

8. **Proof of the theorem.** Let Q be a reals group in L and let $q\neq e$ be an element of Q. Then by Lemma 6.3

$$L[q]\to L/L_{Q^+}=S$$

is a nontrivial covering space of S. However, since S is simply connected, it cannot have a nontrivial covering and this contradiction shows that Assumption (B) is false. As we have seen this proves the theorem.

Bibliography

1. D. Montgomery and L. Zippin, *Existence of subgroups isomorphic to the real numbers*, Ann. of Math. vol. 53 (1951).

2. D. Montgomery, *Connected one dimensional groups*, Ann. of Math. vol. 49 (1948) pp. 110–117.

3. ———, *Dimension of factor spaces*, Ann. of Math. vol. 49 (1948) pp. 373–378.

4. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, 1941.

5. D. Montgomery and L. Zippin, *Translation groups of three space*, Amer. J. Math. vol. 59 (1937) pp. 121–128.

6. K. Iwasawa, *On some types of topological groups*, Ann. of Math. vol. 50 (1949) pp. 507–557.

7. J. von Neumann, *Die Einführung analytischer Parameter in topologischen Grup-*

pen, Ann. of Math. vol. 34 (1933) pp. 170–190.

8. L. Pontrjagin, *Topological groups*, Princeton University Press, 1939.

9. S. Kakutani, *Über die Metrisation der topologischen Gruppen*, Proc. Imp. Acad. Tokyo vol. 12 (1936) pp. 82.

INSTITUTE FOR ADVANCED STUDY AND
QUEENS COLLEGE

A NOTE OF CORRECTION

EDWIN E. MOISE

It has been pointed out to me by R. H. Bing that the proof of Theorem 1 of my paper on the Menger convexification problem[1] is erroneous; Lemma 2 of this argument is false. In the meantime, Theorem 2 of my paper (which depended essentially on Theorem 1) has been proved by Bing.[2] Theorem 2 is therefore a valid foundation for the rest of my argument for the convexification theorem.

UNIVERSITY OF MICHIGAN AND
INSTITUTE FOR ADVANCED STUDY

Received by the editors January 31, 1951.

[1] *Grille decompositions and convexification theorems for compact metric locally connected continua*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1111–1121.

[2] *Complementary domains of continuous curves*, Fund. Math. vol. 36 (1949) pp. 303–318; Theorem 8, p. 308.