ON A QUESTION RAISED BY GARRETT BIRKHOFF

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Birkhoff[1] proposes the problem: Determine the largest integer, k, such that each group whose order is the product of k primes (not necessarily distinct) has subgroup lattice of length k. The condition is satisfied by a group, G, if and only if G contains a sequence of subgroups,

$$G, G_1, G_2, \cdots, G_k = \text{identity},$$

each a subgroup of prime index in the preceding. He outlines a proof[2] that each group with $k \leqq 4$ has this property. An example will now be given of a group, with $k=5$, which does not satisfy the condition, since it has no subgroup of prime index.

There exists a simple group,[3] G, of order $2^2 \cdot 3 \cdot 7 \cdot 13$.

G has no subgroup of index 2, for such a subgroup would be normal[4] in G, in contradiction to the fact that G is simple.

G has $2 \cdot 3 \cdot 13$ Sylow subgroups[5] of order 7, for no other divisor (>1) of $2^2 \cdot 3 \cdot 13$ is $\equiv 1 (\text{mod } 7)$; and the normalizer[5] in G of each Sylow subgroup of order 7 is of order $2 \cdot 7$. A subgroup, H, of index 3 or 13 in G could have only one Sylow subgroup of order 7, necessarily normal in H of order respectively $2^2 \cdot 7 \cdot 13$ or $2^2 \cdot 3 \cdot 7$, both of which are larger than $2 \cdot 7$. Hence, G has no subgroup of index 3 or 13.

G has $2 \cdot 7$ Sylow subgroups of order 13; the normalizer in G of each is of order $2 \cdot 3 \cdot 13$. A subgroup, M, of index 7 in G could have only one Sylow subgroup of order 13, normal in M of order $2^2 \cdot 3 \cdot 13 > 2 \cdot 3 \cdot 13$. And G has no subgroup of index 7.

In the direct product of G above and any solvable group, each subgroup of prime index must contain G. Hence, for each integer $n \geqq 5$, there exists a group with n factors in its order not satisfying the condition.

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[1] Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, rev. ed., Problem 39, p. 99.

[2] Ibid. p. 98.

[3] Burnside, *Theory of groups of finite order*, Cambridge University Press, 1897, pp. 338, 367.

[4] Speiser, *Die Theorie der Gruppen von endlicher Ordnung*, New York, Dover, Theorem 14, p. 30.

[5] Ibid. Theorem 74, p. 67.