INTERIOR MAPPING OF AN ORIENTABLE SURFACE INTO S^2

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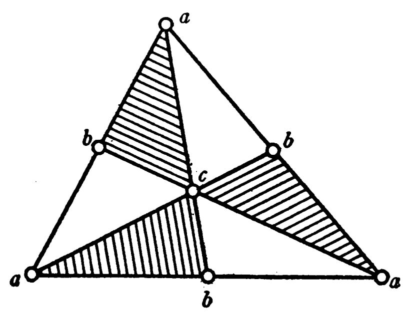
1. An important theorem of Stoilow [2, p. 76][1] states that an orientable surface (surface = 2-dimensional separable manifold[2]) admits an interior mapping into S^2. This result is of significance in the topological classification of Riemann surfaces inasmuch as it can be used as a fundamental part of the proof that the topological surfaces which may be rendered Riemann surfaces are precisely the orientable ones.



However the proof given by Stoilow for his theorem has been found unconvincing by a number of mathematicians (it is not clear how an admissible transition from the nth to the $(n+1)$st stage of the construction is guaranteed). In this note we communicate a short proof of the Stoilow theorem which is free from this objection. Actually we prove more:

Every orientable surface admits an interior mapping onto S^2 which is ramified over precisely three points and has the property that the components of the antecedent of each sufficiently small region of S^2 are relatively compact.

In fact, the ramification indices for two of the distinguished points may be taken as 1 and 2 respectively.

2. We consider a triangulation of the given surface and assume that each 2-cell of the triangulation is barycentrically subdivided in the manner indicated by the accompanying figure. Since the surface

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] We recall Radó's theorem: A surface is triangulable [1].

is orientable, it is possible to assign unambiguously to each of the 2-cells of the barycentric subdivision of the given triangulation one of the designations "shaded," "unshaded" in such a manner that no two distinct 2-cells of the same designation have an edge in common. This may be achieved as follows. Each 2-cell of the subdivision has precisely one edge which lies on a 1-cell of the original triangulation. If this edge contains the "initial" point of the 1-cell where "initial" is taken in the sense of the orientation of the 2-cell of the original triangulation which contains the given 2-cell of the subdivision, we term the 2-cell of the subdivision "shaded," otherwise "unshaded."

We let α, β, γ denote three distinct points of the equator of S^2, $\alpha\beta$ the arc free from γ, and so on, and map the 1-cells ab homeomorphically onto $\alpha\beta$ so that a goes into α and b into β, and proceed similarly for the 1-cells bc and ca with $\beta\gamma$ and $\gamma\alpha$ respectively. On each "shaded" 2-cell the mapping can be extended to a homeomorphic mapping of the 2-cell onto the closed northern hemisphere and similarly on each "unshaded" 2-cell the mapping can be extended to a homeomorphic mapping of the 2-cell onto the closed southern hemisphere. The resulting map of the surface onto S^2 fulfills all the imposed conditions.

Bibliography

1. T. Radó, *Ueber den Begriff der Riemannschen Fläche*, Acta Univ. Szeged. vol. 2 (1925) pp. 101–121.

2. S. Stoilow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, Paris, 1938.

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