Theorems 1 and 2 have their analogues in the case of integrated Lipschitz condition; that is, the following theorem holds:

THEOREM 3. *If* $f(x) \in \mathrm{Lip}\,(\alpha, p)$, $0<\alpha<1$, $p \geqq 1$, *then for any* $\beta>\alpha$

$$\left(\int_0^1 \left| \sigma_n^{(\beta)}(x; f) - f(x) \right|^p dx\right)^{1/p} = O(n^{-\alpha}). \tag{25}$$

Since the proof is analogous to the preceding one, we shall omit it.

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POSITIVE INFINITIES OF POTENTIALS

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Let R denote Euclidean 3-space. The following theorem is due to Evans [1, p. 421].[1]

Let E be a closed bounded set of capacity zero in R. There exists a distribution of positive mass $\mu(e)$ entirely on E, such that its potential $V(M) = \int_R (1/MP)\, d\mu(P)$ is infinite at every point of E and at no other points.

A proof of the two-dimensional analogue was published by Noshiro [2]. In the present note we show, by a modification of Evans' construction, that an *absolutely continuous* distribution exists whose potential is infinite on the preassigned set E only. More precisely, our result, extended to unbounded sets, is as follows:

THEOREM. *Let E be a closed set of capacity zero in R, and let G be an open set containing E. Then there exists a non-negative function f which is summable on R, such that the superharmonic function (that is, the potential)*

$$F(M) = \int_R \frac{1}{MP} f(P)dP$$

is infinite on E, is continuous in $R-E$, and is harmonic in $R-\overline{G}$. ($\overline{G}$ denotes the closure of G.)

Analogous results evidently hold in two and in more than three dimensions.

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[1] Numbers in brackets refer to the references at the end of the paper.

To begin with, we suppose that E is bounded and contains infinitely many points. Let n be a positive integer, and put

$$(1)\qquad V_n = \max_{Q_1,\cdots,Q_n\in E}\left\{\min_{P\in E}\frac{1}{n}\left(\frac{1}{PQ_1}+\cdots+\frac{1}{PQ_n}\right)\right\}.$$

Since E contains infinitely many points, $V_n<+\infty$. The compactness of E implies that there exist points $P_1,\cdots,P_n$ on E, such that, for all P on E,

$$(2)\qquad \frac{1}{n}\left(\frac{1}{PP_1}+\cdots+\frac{1}{PP_n}\right)\geqq V_n.$$

The transfinite diameter of E is defined as the limit of the sequence $\{D_n\}$ where

$$\frac{n(n-1)}{2}\cdot\frac{1}{D_n} = \min_{Q_1,\cdots,Q_n\in E}\left\{\sum_{1\leqq i<k\leqq n}\frac{1}{Q_iQ_k}\right\}.$$

It can be shown [1, p. 423] that $V_n\geqq(D_{n+1})^{-1}$. The transfinite diameter of a compact set being equal to its capacity [3], it follows that

$$(3)\qquad \lim_{n\to\infty} V_n = +\infty.$$

So far we have followed Evans. We now choose r_n such that $0<r_n<(nV_n)^{-1}$ (this is possible, since $V_n<+\infty$), and such that the closed spheres S_i with centers at P_i $(i=1,\cdots,n)$ and radius r_n are contained in G. For $i=1,\cdots,n$, we define

$$\phi_i(P) = \begin{cases}\dfrac{3}{4n\pi r_n^3} & (P\in S_i),\\ 0 & (P\in R-S_i),\end{cases}\qquad u_n(P)=\sum_{i=1}^{n}\phi_i(P)\qquad (P\in R),$$

$$U_n(M)=\int_R\frac{1}{MP}u_n(P)dP\qquad (M\in R).$$

Then $\int_R u_n(P)dP=1$, and U_n is the potential of a unit mass. Suppose $M\in E$. If $M\in R-(S_1+\cdots+S_n)$, then

$$(4)\qquad U_n(M)=\frac{1}{n}\left(\frac{1}{MP_1}+\cdots+\frac{1}{MP_n}\right)\geqq V_n$$

(by (2)). If $M\in S_j$, then

$$(5)\qquad U_n(M)>\int_{S_j}\frac{1}{MP}\phi_j(P)dP=\frac{3r_n^2-t^2}{2nr_n^3}\geqq\frac{1}{nr_n}>V_n,$$

where $t=MP_j$. By (4), (5),

$$U_n(M) \geqq V_n \qquad (M \in E). \tag{6}$$

By (3), there is a sequence $\{n_k\}$ such that $V_{n_k} \geqq 2^k$. We define

$$f(P) = \sum_{k=1}^{\infty} 2^{-k} u_{n_k}(P) \qquad (P \in R),$$

$$F(M) = \int_R \frac{1}{MP} f(P)dP = \sum_{k=1}^{\infty} 2^{-k} U_{n_k}(M) \qquad (M \in R).$$

Then $\int_R f(P)dP = 1$; by (6), $F(M) = +\infty$ if $M \in E$; if $M \in R-E$, f is bounded in some neighborhood of M (since $r_n \to 0$ as $n \to \infty$), which implies that F is continuous in $R-E$; and in $R-\overline{G}$, $f=0$, hence F is harmonic.

Next, if E is finite, let $E = A_1 + \cdots + A_m$. Choose $r > 0$ such that the closed spheres S_i with centers at A_i $(i=1, \cdots, m)$ and radius r are contained in G, and define

$$\phi(t) = \begin{cases} t^{-2} & \text{if } 0 < t < r, \\ 0 & \text{otherwise}, \end{cases} \qquad f(P) = \sum_{i=1}^{m} (\phi PA_i) \qquad (P \in R).$$

The conclusion of the theorem evidently holds for this function f. Hence the theorem is proved for bounded sets E.

Finally, suppose E is unbounded. There exist compact sets E_i $(i=1, 2, 3, \cdots)$ such that $R = \sum_{i=1}^{\infty} E_i$, and open sets G_i containing E_i such that no point of R is in more than four of the sets G_i. We now apply the previously obtained result for bounded sets to construct functions f_i $(i=1, 2, 3, \cdots)$ which satisfy the conclusion of the theorem with respect to the sets $E \cdot E_i$ and $G \cdot G_i$, such that $\int_R f_i(P)dP = 2^{-i}$, and put $f(P) = \sum_{i=1}^{\infty} f_i(P)$. For any P, this sum contains at most four nonzero terms. Hence it is easily verified that the conclusion of the theorem holds.

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