CERTAIN HOMOGENEOUS UNICOHERENT INDECOMPOSABLE CONTINUA

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A simple closed curve is the simplest example of a compact, nondegenerate, homogeneous continuum. If a bounded, nondegenerate, homogeneous plane continuum has any local connectedness property, even of the weakest sort, it is known to be a simple closed curve [1, 2, 3].[1] The recent discovery of a bounded, nondegenerate, homogenous plane continuum which does *not* separate the plane [4, 5] has given substance to the old question as to whether or not such a continuum must be indecomposable. Under certain conditions such a continuum must *contain* an indecomposable continuum [6]. It is the main purpose of this paper to show that every bounded homogeneous plane continuum which does not separate the plane *is* indecomposable.

NOTATION. If M is a continuum and x is a point of M, U_x will be used to denote the set of all points z of M such that M is aposyndetic at z with respect to x.[2] It is evident that U_x is an open subset of M.

LEMMA. *If the compact metric continuum M is homogeneous and x and y are distinct points of M, then U_y is not a proper subset of U_x.*

PROOF. Suppose on the contrary that U_y is a proper subset of U_x. Since M is homogeneous, there exists a homeomorphism T such that $T(M)=M$ and $T(x)=y$. Then $T(U_x)=U_y$ and $T(U_y)$ is $U_{T(y)}$ which is a proper subset of U_y. Hence there exists a sequence $x_0=x$, $x_1=y$, $x_2=T(y), \cdots, x_n=T^n(x), \cdots$ of points of M such that for each positive integer n, U_{x_n} is a proper subset of $U_{x_{n-1}}$. For no two nonnegative integers i and j is $x_i=x_j$, because if $x_i=x_j$ then $U_{x_i}=U_{x_j}$. Consequently the sequence $x_1, x_2, x_3, \cdots$ has a limit point x_ω. Now for each positive integer n, U_{x_ω} is a subset of U_{x_n}, because if p is a point of U_{x_ω} there exist a subcontinuum K of M and an open subset V of M such that $M-x_\omega \supset K \supset V \supset p$; hence for infinitely many positive integers n, $M-x_n \supset K \supset V \supset p$; so for infinitely many positive integers n, M is aposyndetic at p with respect to x_n and hence p belongs to U_{x_n}.

Evidently $x_\omega \neq x_n$, $n=1, 2, 3, \cdots$. And since M is homogeneous,

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[1] Numbers in brackets refer to the bibliography at the end of this paper.

[2] The continuum M is *aposyndetic* at the point z of M with respect to the point x of M provided that M contains a continuum K and an open (rel. M) subset V such that $M-x \supset K \supset V \supset z$.

there exists a homeomorphism T_1 such that $T_1(M)=M$ and $T_1(x)=x_\omega$. Then $T_1TT_1^{-1}$ is a homeomorphism of M onto itself such that if we let $x_{\omega+1}=T_1TT_1^{-1}(x_\omega)$, $T_1TT_1^{-1}(U_{x_\omega})=U_{x_{\omega+1}}$ which is a proper subset of U_{x_ω}. This process can be continued *uncountably* many times to produce a well-ordered sequence $\alpha=x_1, x_2, x_3, \cdots, x_i, \cdots$ $(i<\omega_1)$, of distinct points of M such that (1) if x_i of α has no immediate predecessor in α, x_i is a limit point of some countable subsequence of α running through the terms of α preceding x_i in α, and (2) $U_{x_1}, U_{x_2}, U_{x_3}, \cdots, U_{x_i}, \cdots$ is a monotone descending sequence of distinct open subsets of M. In a compact metric space (2) is impossible.

THEOREM 1. *A homogeneous, hereditarily unicoherent, compact metric continuum M is indecomposable.*

PROOF.[3] Suppose that U is an open subset of M and H is a subset of $M-U$ such that in order for a point x to belong to H it is necessary and sufficient that $U_x=U$. In case M contains no such sets U and H, M is indecomposable by Theorem 9 of [**7**].

It is rather easy to see that H is closed. Suppose that there exists a point y of $\overline{H}-H$. Let z be a point of U_y. Then M is aposyndetic at z with respect to y and hence M is aposyndetic with respect to some point of H. Consequently z belongs to U. But by the lemma U_y cannot be a proper subset of U and hence $U_y=U$ and y belongs to H. So H is closed.

If w is a point of M such that some point x of H cuts w from a point z_1 of U, then x cuts w from all points of U and w belongs to H.[4] For suppose that z_2 is a point of U. There exist continua K_1 and K_2 and open sets V_1 and V_2 such that $M-x\supset K_i\supset V_i\supset z_i$ $(i=1, 2)$. Now if x cuts V_1 from V_2, it follows from the homogeneity of M that *every* point of M cuts between two open subsets of M; but by Corollary 2 of [**8**], this is impossible. So x does not cut V_1 from V_2 and hence there exists a continuum K in $M-x$ such that $K\cdot V_1\neq 0$ and $K\cdot V_2\neq 0$. The continuum K_1+K+K_2 contains z_1 but not x; hence K_1+K+K_2 does not contain w; consequently x cuts w from all points z_2 of U and furthermore z_2 belongs to U_w. This shows that U is a subset of U_w and, by the lemma, $U=U_w$. So w belongs to H.

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[3] Throughout this proof M will be considered to be space. If there do not exist points x and z of M such that M is aposyndetic at z with respect to x, then M is indecomposable (Theorem 9 of [**7**]). So because M is homogeneous it will be assumed that for each point x of M, U_x exists (that is, nonvacuous).

[4] A point x *cuts* w from z (in M) provided that there exists no subcontinuum of M lying in $M-x$ and containing $w+z$.

For each point o of H, let N_o denote o together with all points x of H such that x cuts o from U. The set N_o is closed. Now suppose that for some point o of H, o does not cut all other points of N_o from U. Then N_o contains a point o_1 such that N_{o_1} is a subset of $N_o - o$. A homeomorphism of M onto itself carrying o into o_1 leaves U invariant and carries o_1 into a point o_2 of H such that N_{o_2} is a proper subset of N_{o_1}. As in the proof of the lemma, this process may be continued uncountably many times to produce an uncountable monotone sequence of distinct closed sets. This is impossible. Consequently o cuts all other points of N_o from U. It follows at once that each point of N_o cuts all other points of N_o from U and in particular if a point p of H cuts a point o of H from U, then o cuts p from U, and $N_o = N_p$.

The set H contains no domain.[5] For suppose on the contrary that H contains a domain D. Let o denote a point of H. Then N_o does not contain D, for if it did, a point x of D could cut the domain $D-x$ from the domain U contrary to Corollary 2 of [8]. So $D-D \cdot N_o$ is a domain in H containing no point of N_o. Now M is not aposyndetic at any point of $D-D\cdot N_o$ with respect to a point of N_o. Hence by Theorem 6 of [7], if z is a point of U, $D-D\cdot N_o$ contains a point x and N_o contains a point y such that y cuts x from z and hence from U. Therefore y cuts x from U and consequently x belongs to N_o. This is a contradiction since x belongs to $D-D\cdot N_o$. So H contains no domain.

The domain U is dense in M. Suppose the contrary. There exists a domain D lying in $M-(\overline{U}+H)$. Let y be a point of H. By the definition of U, M is *not* aposyndetic at any point of D with respect to y. Let z be a point of U. By Theorem 6 of [7], D contains a point x such that y cuts x from z. Hence (by paragraph 3 of this proof) x belongs to H contrary to construction. So U is dense in M and the boundary of U is $M-U$.

The set $M-U$ is a continuum. Obviously $M-U$ is closed. Suppose that $M-U$ is not connected; then $M-U=A+B$ where $\overline{A}=A$, $\overline{B}=B$, and $A\cdot B=0$. Suppose that A contains a point x of H. There exists a domain D such that D contains B but $\overline{D}\cdot A=0$. Each point of the boundary β of D belongs to U; so there exist a finite collection $K_1, K_2, \cdots, K_n$ of continua and a collection $V_1, V_2, \cdots, V_n$ of domains such that $V_1, V_2, \cdots, V_n$ covers β and for each i, $1 \leqq i \leqq n$, $M-x \supset K_i \supset V_i$. Since by Corollary 2 of [8] (and the homogeneity of M) x does not cut any two domains from each other, there exists a

[5] An open subset of M is called a *domain*. A domain is not necessarily connected as used here.

continuum K in $M-x$ which contains D. Hence M is aposyndetic at each point of B with respect to x. This is contrary to the definitions of B and U. Hence $M-U$ is connected.

Let o be a point of H. Then $N_o=M-U$. Suppose on the contrary that q is a point of $M-U$ not in N_o. If q cuts o from a point of U_q, then q cuts U_q from o. Let T be a homeomorphism of M onto itself carrying o into q.[6] Evidently $T(U)=U_q$ and (by paragraph 3, $T(H)$ taking the role of H) o belongs to $T(H)$. Therefore o cuts q from U_q. But $U_q \cdot U \neq 0$ since both U and U_q are open, dense subsets of M; so o cuts q from a point of U. Hence q belongs to H. It follows that q cuts o from U and thus belongs to N_o. From this contradiction it is evident that no point q of $M-(U+N_o)$ cuts o from a point of U_q. Now let K be a continuum containing a domain V of U and lying in $M-o$. Since each point of N_o cuts every other point of N_o from U, K contains no point of N_o. Since no point of N_o cuts a point q of $M-(U+N_o)$ from a point of U, K may be assumed to contain a point of $M-(U+N_o)$. For each point q of $K\cdot[M-(U+N_o)]$ there exists a continuum C_q from V to o lying in $M-q$ ($V\cdot U_q \neq 0$). Let F denote a finite collection of these continua, C_q, such that if p is a point of $K\cdot[M-(U+N_o)]$, some element of F lies in $M-p$. Suppose that some two continua C_q and C_t of F intersect in a continuum C containing no point of K. Then C_q+K and C_t+K are continua whose intersection is $C+K$ which is not a continuum. Since M is hereditorily unicoherent, this is a contradiction. Hence C contains a point of K. Because M is hereditarily unicoherent, the same reasoning holds when we suppose that C is the common part of all elements of F. In this case $C\cdot K$ contains no point of $M-U$ and hence is a subset of U. Then $(M-U)+C$ and $(M-U)+K$ are continua whose common part is $(M-U)+C\cdot K$ which is not connected. So $N_o=M-U$.

Thus H is a continuum; H is the boundary of U; $U+H=M$; and every point of H cuts every other point of H from U. Let G be a collection consisting of H together with every image of H under homeomorphisms of M onto itself. It is easy to see that G fills up M and no two elements of G have a point in common. Furthermore, G is upper-semicontinuous for if some sequence $x_1, x_2, x_3, \cdots$ of points of distinct elements of G converged to a point x of an element of G, say H, but some infinite sequence $y_1, y_2, y_3, \cdots$ of points from the same elements of G converged to a point y of $M-H$, then M would be aposyndetic at y with respect to x. But for each i, M is not

[6] Roughly stated the purpose of T is merely to shift the frame of reference from o to q, so that results already obtained for H and U will apply to similar sets constructed for q.

aposyndetic at y_i with respect to x_i, and this contradicts Theorem 1 of [**7**]. So G is upper-semicontinuous.

With respect to its elements as points, G is a continuum M'. Furthermore M' is homogeneous and aposyndetic. In such a continuum the meaning of "cut point" and "separating point" are the same [**9**]. Since M' contains a nonseparating point, every point of M' is a nonseparating point because of the homeogeneity. Let A and B denote distinct points of M' and let T denote a continuum in M' irreducible from A to B. Let X denote a point of $T-(A+B)$. There exists in $M'-X$ a continuum T_1 containing $A+B$. But M' is hereditarily unicoherent. So $T \cdot T_1$ is a subcontinuum of T containing $A+B$ but not X. This is a contradiction and from this contradiction Theorem 1 follows.

THEOREM 2. *If M is a homogeneous, bounded, plane continuum which does not separate the plane, M is indecomposable.*

Theorem 2 follows immediately from Theorem 1.

The following question remains unanswered: Is every homogeneous, bounded, nondegenerate, plane continuum which does not separate the plane a pseudo-arc?

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