the resulting 2-cell be amalgamated with α_2 across c_2 to obtain a 2-cell ρ with p for boundary. By Theorem 2.1, ρ is the interior of p. Since α_1 and α_2 are exterior to c, the 2-cell α constitutes the entire interior of c. The Jordan-Schoenflies Theorem now follows readily in all its generality.

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A NOTE ON CURVATURE AND BETTI NUMBERS

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1. S. Bochner has proved the following theorem [2]:[1] Let $M^{(m)}$ be a closed manifold with complex structure [4; 7] of complex dimension m, on which there exists a Kähler-metric [2; 3; 5][2]

$$ds^2 = g_{ik^*}(dz^i dz^*),^3 \tag{1}$$

$$\frac{\partial g_{ik^*}}{\partial z_l} = \frac{\partial g_{lk^*}}{\partial z_i}. \tag{2}$$

Let R_{ik^*} denote the Ricci tensor and

$$P_{hi^*jk^*} = R_{hi^*jk^*} - \frac{1}{m+1}(g_{hi^*}R_{jk^*} + g_{hk^*}R_{i^*j}) \tag{3}$$

the tensor of projective curvature. In every point of $M^{(m)}$ we form the numbers

$$L = \inf_{\xi} \frac{-R_{ik^*}\xi^i\xi^{k^*}}{\xi^i\xi_i}, \tag{4}$$

$$P = \sup_{\xi} \left| \frac{P_{hi^*jk^*}\xi^{hi^*}\xi^{jk^*}}{\xi^{hi^*}\xi_{hi^*}} \right|, \tag{5}$$

with all vectors ξ^i and skew-symmetric tensors ξ^{ij^*} attached to the point in question. If

$$L > 0 \tag{6}$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Products of differentials in parentheses denote ordinary products, products without parentheses are skew products.

[3] We denote by i^* the index relative to z^{*i}.

holds everywhere on $M^{(m)}$, and if, for some p, we have at all points the relation

$$(p-1)P < \left(1 - \frac{p-1}{m+1}\right)L, \tag{7}$$

then for the Betti numbers p^q of $M^{(m)}$ we have

$$p^{2k} = 1, \qquad 2k \leqq p, \tag{8}$$

$$p^{2k+1} = 0, \qquad 2k+1 \leqq p. \tag{9}$$

2. The proof of this theorem in [2] is based on Bochner's lemma [1]:

If on a compact (differentiable) manifold we have, for a given scalar Ψ, $\Delta\Psi \geqq 0$ everywhere, then $\Delta\Psi = 0$ everywhere.

By the theory of harmonic integrals [5, §2] the group of complex harmonic differential forms of degree q on $M^{(m)}$ is isomorphic to the qth cohomology group with real coefficients of $M^{(m)}$. The form

$$\Omega = g_{ik^*} dz^i dz^{*k} \tag{10}$$

with all its powers Ω^k, $k \leqq m$, is harmonic and not equal to 0 [3; 5; 6]. It is known that every harmonic differential form ϕ^q of degree $q \leqq m$ is a sum with constant coefficients c_i,

$$\phi^q = \sum_{i=0}^{[q/2]} c_i \chi^{q-2i} \Omega^i, \tag{11}$$

where the forms χ are harmonic and "effective," that is, satisfy the condition

$$*\Omega * \chi = 0. \tag{12}$$

A differential form $\phi^q_{(h)}$ is said to be *pure*, and *of type h*, if it is a homogeneous form of degree h in the differentials dz^{*i}. Every harmonic form is a sum of pure harmonic forms. Let $p^q_{(h)}$ be the rank of the linear space of the pure harmonic forms on $M^{(m)}$ of degree q and type h. Then

$$p^{2k} \equiv p^{2k}_{(k)} \pmod 2, \tag{13}$$

$$p^{2k+1} \equiv 0 \pmod 2. \tag{14}$$

(14) has been proved first by Lefschetz in the case of algebraic manifolds.

If ϕ^q is a pure form $\phi^q_{(h)}$, then (11) becomes [5]

$$\phi^{q}_{(h)} = \sum c_i \chi^{q-2i}_{(h-i)} \Omega^{i}. \tag{11'}$$

3. We now prove the following theorem.

THEOREM. *If on $M^{(m)}$ we have* (6) *and* (7) *everywhere, then there exist no effective harmonic forms of type k and degree $2k$ for $k \leqq p$, that is, we have*

$$p^{2k} \equiv 1 \pmod 2, \qquad 2k \leqq 2p. \tag{15}$$

From (8), (9), (14), and (15) we see that we have the following situation if (6) and (7) hold everywhere on $M^{(m)}$:

$$\begin{array}{llll} p^{2k} = 1 \quad \text{for} \quad 2k \leqq p, & p^{2k} \equiv 1 \pmod 2 \quad \text{for} \quad 2k \leqq 2p, \\ p^{2k+1} = 0 \quad \text{for} \quad 2k+1 \leqq p, & p^{2k} \equiv 0 \pmod 2 \quad \text{always}. \end{array}$$

PROOF. It is a fundamental property of a Kähler metric [5] that only those components of the curvature tensor which are equal to one of the form $R_{hj^*kl^*}$ are not equal to 0. Let $\psi^{q}_{(k)} = P_{i_1 \cdots i_{q-k} i^*_{q-k+1} \cdots i^*_q} dz^{i_1} \cdots dz^{*i_q}$ be an effective form of degree q and type k. Then (12) is

$$g^{r s^*} P_{r i_2 \cdots i_q s^* i^*_{q-k+2} \cdots i^*_q} = 0. \tag{16}$$

Let $\Psi(\psi^{q}_{(k)})$ be the scalar $P_{i_1 \cdots i^*_q} P^{i_1 \cdots i^*_q}$ and $k \leqq q-k$. Condition (16) gives [2, (36)]

$$\begin{aligned} \frac{1}{2}\Delta\Psi = {} & P_{i_1 \cdots i^*_q, \lambda} P^{i_1 \cdots i^*_q, \lambda} \\ & + \frac{1}{k}\{(k-1) P_{s i_2 \cdots i_{q-k} t^* i^*_{q-k+2} \cdots i^*_q} P^{j_1 i_2 \cdots i_{q-k} j^*_2 i^*_{q-k+2} \cdots i^*_q} P^{st^*}_{j_1 j_2} \\ & - \left(1 - \frac{k-1}{m+1}\right) P_{i_1 \cdots s^* \cdots i^*_q} P^{i_1 \cdots t^* \cdots i^*_q} R^{s^*}_{t^*}. \end{aligned}$$

This formula shows that from (6) and (7) follows

$$\Delta\Psi(\psi^{2k}_{(k)}) > 0, \qquad k \leqq p,$$

a contradiction to Bochner's lemma.

4. The theory of effective harmonic forms and (11) is not restricted to Kähler manifolds [5]. In fact, if on a Riemannian manifold $M^{(m)}$ of dimension $2m$ a 2-form exists,

$$\Omega = h_{ik} dx^i dx^k,$$

which is closed and is everywhere of the same rank 2ρ, then (11) is valid for $q \leqq \rho$, the effective forms being defined analogously to (12) as forms $\phi^q = F_{i_1 \cdots i_q} dx^{i_1} \cdots dx^{i_q}$ satisfying

$$h^{ik} F_{iki_3 \cdots i_q} = 0.$$

Let P_{hijk} be defined as

$$P_{hijk} = R_{hijk} - \frac{1}{m+1} (g_{hi} R_{jk} + h_{hk} R_{ij}).$$

It is easy to see that (6) and (7) imply (8) and (9) for $2k$ resp. $2k+1 \leqq \min (p, \rho)$.

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