

THE EXTENDED CENTRALIZER OF A RING OVER A MODULE

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In a recent paper,¹ K. Asano gave a new proof of the theorem that a domain of integrity has a right quotient ring if and only if every pair of nonzero elements has a common nonzero right multiple. His method of proof is used in the present work to extend the centralizer of a ring over a module to a system of semi-endomorphisms of the module. From this extension, necessary and sufficient conditions that a ring have a right quotient regular ring are derived.

Consider a given ring R , and a given nonzero right R -module M . Denote by \mathfrak{M} the set of all submodules of M , and by \mathfrak{M}^* the set of all submodules N of M having the property that $N \cap N' \neq 0$ for all nonzero $N' \in \mathfrak{M}$. Since $M \in \mathfrak{M}^*$, \mathfrak{M}^* is not void. It is easily seen that if N and N' are in \mathfrak{M}^* , then $N + N'$ and $N \cap N'$ are also in \mathfrak{M}^* . Thus $\{\mathfrak{M}^*; \subseteq, \cap, +\}$ is a sublattice of the lattice $\{\mathfrak{M}; \subseteq, \cap, +\}$.

An R -homomorphism of N into M , N any element of \mathfrak{M} , is called a *semi-endomorphism* of M . Thus, thinking of the semi-endomorphism α as a left operator on N , we have $\alpha(x+y) = \alpha x + \alpha y$ and $\alpha(xa) = (\alpha x)a$ for all $x, y \in N, a \in R$. For convenience, the module N on which α is defined is denoted by M_α .

The set of all semi-endomorphisms of M is labeled with \mathfrak{A} . Contained in \mathfrak{A} is the usual centralizer of R over M consisting of all $\alpha \in \mathfrak{A}$ for which $M_\alpha = M$. A partial ordering \leq is defined in \mathfrak{A} as follows: $\alpha \leq \beta$ if and only if $M_\alpha \subseteq M_\beta$ and $\alpha x = \beta x$ for all $x \in M_\alpha$. The notation $\alpha < \beta$ is used in case $\alpha \leq \beta$ and $M_\alpha \neq M_\beta$.

In case \mathfrak{L} is a linearly ordered subset of \mathfrak{A} , and $M' = \bigcup M_\alpha, \alpha \in \mathfrak{L}$, the mapping γ of M' into M defined by

$$\gamma x = \alpha x \quad \text{whenever} \quad x \in M_\alpha, \alpha \in \mathfrak{L},$$

is easily verified to be an element of \mathfrak{A} such that $\gamma \geq \alpha$ for all $\alpha \in \mathfrak{L}$. Thus, by Zorn's Lemma, every α of \mathfrak{A} is contained in a maximal element of \mathfrak{A} . Let \mathfrak{B} denote the set of all maximal elements of \mathfrak{A} . Obviously the centralizer of R over M is contained in \mathfrak{B} . For any $\beta \in \mathfrak{B}$, $M_\beta \in \mathfrak{M}^*$. Otherwise there would exist a nonzero $N \in \mathfrak{M}$ such that $N \cap M_\beta = 0$, and the semi-endomorphism α defined by

$$\alpha x = \beta x, \quad x \in M_\beta; \quad \alpha x = 0, \quad x \in N,$$

would exceed β .

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¹ Journal of the Mathematical Society of Japan vol. 1 (1949) pp. 73-78.

For any $\alpha, \beta \in \mathfrak{A}$, define $M_\beta^\alpha \in \mathfrak{M}$ by

$$M_\beta^\alpha = \{x \mid x \in M_\beta, \beta x \in M_\alpha\}.$$

Observe that if $M_\alpha, M_\beta \in \mathfrak{M}^*$, then also $M_\beta^\alpha \in \mathfrak{M}^*$. For if $N \in \mathfrak{M}$, $N \neq 0$, then $N \cap M_\beta \neq 0$; if $\beta(N \cap M_\beta) \neq 0$, then $\beta(N \cap M_\beta) \cap M_\alpha \neq 0$, $(N \cap M_\beta) \cap M_\beta^\alpha \neq 0$, and therefore $N \cap M_\beta^\alpha \neq 0$; if, on the other hand, $\beta(N \cap M_\beta) = 0$, then $N \cap M_\beta \subseteq M_\beta^\alpha$ and again $N \cap M_\beta^\alpha \neq 0$.

Operations of addition and multiplication are defined in \mathfrak{A} in the obvious way. Thus for $\alpha, \beta \in \mathfrak{A}$, $\alpha + \beta$ and $\alpha\beta$ are defined as follows:

$$(\alpha + \beta)x = \alpha x + \beta x, \quad x \in M_\alpha \cap M_\beta; \quad (\alpha\beta)x = \alpha(\beta x), \quad x \in M_\beta^\alpha.$$

By definition, $M_{\alpha+\beta} = M_\alpha \cap M_\beta$ and $M_{\alpha\beta} = M_\beta^\alpha$.

Associated with any $N \in \mathfrak{M}$ are the trivial semi-endomorphisms 0_N and 1_N defined by: $0_N x = 0$, $1_N x = x$; $x \in N$. Labelling $0_M = 0$ and $1_M = 1$, evidently $0_N \leq 0$ and $1_N \leq 1$ for all $N \in \mathfrak{M}$. For any $\alpha \in \mathfrak{A}$, $-\alpha$ is defined in the usual way; and $\alpha + (-\alpha) = (-\alpha) + \alpha = 0_N$ where $N = M_\alpha = M_{-\alpha}$.

Every $\alpha \in \mathfrak{A}$ that is an isomorphism of M_α into M has an inverse α^{-1} defined by

$$\alpha^{-1}(\alpha x) = x, \quad x \in M_\alpha; \quad M_{\alpha^{-1}} = \alpha M_\alpha.$$

The set of all such isomorphisms contained in \mathfrak{A} is denoted by \mathfrak{U} . It is evident that all $1_N \in \mathfrak{U}$, $N \in \mathfrak{M}$, and whenever $\alpha \in \mathfrak{U}$, also $\alpha^{-1} \in \mathfrak{U}$.

The properties enjoyed by the operations in \mathfrak{A} are summarized in the following theorem.

THEOREM 1. *The algebraic system $\{\mathfrak{A}; +, \cdot, \leq\}$ has the following properties:*

(1) $\{\mathfrak{A}; +\}$ is an abelian semigroup with identity element 0. Associated with each $\alpha \in \mathfrak{A}$ are unique elements $-\alpha$ and 0_α in \mathfrak{A} such that

$$(i) \quad \alpha + (-\alpha) = 0_\alpha, \quad (ii) \quad \alpha + 0_\alpha = \alpha, \quad (iii) \quad -(-\alpha) = \alpha.$$

(2) $\{\mathfrak{A}; \cdot\}$ is a semigroup with identity element 1.

(3) $\{\mathfrak{U}; \cdot\}$ is a semigroup with identity element 1. Associated with each $\alpha \in \mathfrak{U}$ are unique elements α^{-1} , 1_α , and $1'_\alpha$ in \mathfrak{U} such that

$$(i) \quad \alpha^{-1}\alpha = 1_\alpha, \quad (ii) \quad \alpha\alpha^{-1} = 1'_\alpha, \quad (iii) \quad \alpha 1_\alpha = \alpha,$$

$$(iv) \quad 1'_\alpha \alpha = \alpha, \quad (v) \quad (\alpha^{-1})^{-1} = \alpha.$$

(4) For any $\alpha, \beta, \gamma \in \mathfrak{A}$,

$$(i) (\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha, \quad (ii) \alpha(\beta + \gamma) \geq \alpha\beta + \alpha\gamma.$$

(5) For any $\alpha, \beta, \gamma, \delta \in \mathfrak{A}$ such that $\alpha \leq \beta$ and $\gamma \leq \delta$,

$$(i) \alpha + \gamma \leq \beta + \delta, \quad (ii) \alpha\gamma \leq \beta\delta, \quad (iii) -\alpha \leq -\beta,$$

$$(iv) \alpha^{-1} \leq \beta^{-1} \text{ in case } \alpha, \beta \in \mathfrak{U}.$$

The proofs of (1)–(3) are straightforward, and hence will be omitted. Part (4) is a consequence of the following relations:

$$M_{\alpha}^{\beta+\gamma} = M_{\alpha}^{\beta} \cap M_{\alpha}^{\gamma}, \quad M_{\beta+\gamma}^{\alpha} \supseteq M_{\beta}^{\alpha} \cap M_{\gamma}^{\alpha}.$$

To prove (5) part (iv), assume that $\alpha, \beta \in U$ with $\alpha < \beta$. Then $M_{\alpha} \subset M_{\beta}$, $M_{\alpha} \neq M_{\beta}$, so that $\alpha M_{\alpha} \subset \beta M_{\beta}$, $\alpha M_{\alpha} \neq \beta M_{\beta}$, and hence $\alpha^{-1} < \beta^{-1}$. The proof of the rest of (5) will be omitted.

Let \mathfrak{R} be the subset of \mathfrak{A} containing all α such that $M_{\alpha} \in \mathfrak{M}^*$. The set \mathfrak{R} is closed under the operations of addition and multiplication in view of previous remarks. If in Theorem 1 we replace \mathfrak{U} by $\mathfrak{B} = \{\alpha \mid \alpha, \alpha^{-1} \in \mathfrak{U} \cap \mathfrak{R}\}$, Theorem 1 then applies to the system $\{\mathfrak{R}; +, \cdot, \leq\}$.

For any $\alpha \in \mathfrak{A}$, let $N_{\alpha} = \{x \mid x \in M_{\alpha}, \alpha x = 0\}$, the annihilator of α in M . The subset \mathfrak{S} of \mathfrak{R} defined by

$$\mathfrak{S} = \{\alpha \mid \alpha \in \mathfrak{R}, N_{\alpha} \in \mathfrak{M}^*\}$$

is called the *radical* of \mathfrak{R} . Since $N_{\alpha-\beta} \supseteq N_{\alpha} \cap N_{\beta}$ and $N_{\gamma\alpha} \supseteq N_{\alpha}$, $\alpha - \beta$ and $\gamma\alpha$ are in \mathfrak{S} whenever $\alpha, \beta \in \mathfrak{S}$, $\gamma \in \mathfrak{R}$. That also $\alpha\gamma \in \mathfrak{S}$ for any $\alpha \in \mathfrak{S}$, $\gamma \in \mathfrak{R}$ is seen as follows. If $N \in \mathfrak{M}$, $N \neq 0$, then $N \cap M_{\gamma} \neq 0$. If $\gamma(N \cap M_{\gamma}) = 0$, then $N \cap M_{\gamma} \subseteq N_{\alpha\gamma}$ and $N \cap N_{\alpha\gamma} \neq 0$; while if $\gamma(N \cap M_{\gamma}) \neq 0$, then $\gamma(N \cap M_{\gamma}) \cap N_{\alpha} \neq 0$, $(N \cap M_{\gamma}) \cap N_{\alpha\gamma} \neq 0$, and again $N \cap N_{\alpha\gamma} \neq 0$. We conclude that the radical \mathfrak{S} is an ideal in \mathfrak{R} . Another property of \mathfrak{S} is that if $\beta \in \mathfrak{S}$ and $\alpha, \gamma \in \mathfrak{R}$ with $\alpha \leq \beta \leq \gamma$, then α and γ are in \mathfrak{S} . This is obvious, since $N_{\alpha} = M_{\alpha} \cap N_{\beta}$ and $N_{\gamma} \supseteq N_{\beta}$.

The ideal \mathfrak{S} induces a partition of \mathfrak{R} into the set $\mathfrak{R}/\mathfrak{S}$ of cosets $\bar{\alpha} = \{\beta \mid \beta \in \mathfrak{R}, \alpha - \beta \in \mathfrak{S}\}$, $\alpha \in \mathfrak{R}$. The operations of addition and multiplication can be introduced into $\mathfrak{R}/\mathfrak{S}$ in the usual way, that is, if $\alpha + \beta = \gamma$ and $\alpha\beta = \delta$, then $\bar{\alpha} + \bar{\beta} = \bar{\gamma}$ and $\bar{\alpha}\bar{\beta} = \bar{\delta}$. One easily verifies that $\{\mathfrak{R}/\mathfrak{S}; +\}$ is an abelian group. By Theorem 1, (4), we see that $(\bar{\beta} + \bar{\gamma})\bar{\alpha} = \bar{\beta}\bar{\alpha} + \bar{\gamma}\bar{\alpha}$ and $\bar{\alpha}(\bar{\beta} + \bar{\gamma}) = \bar{\alpha}\bar{\beta} + \bar{\alpha}\bar{\gamma}$. Thus $\{\mathfrak{R}/\mathfrak{S}; +, \cdot\}$ is a ring with a unit element. This ring is called the *extended centralizer* of R over M .

THEOREM 2. *The extended centralizer of R over M is a regular ring.*

To prove this, it is necessary to show that for every $\bar{\alpha} \in \mathfrak{R}/\mathfrak{S}$, there exists $\bar{\beta} \in \mathfrak{R}/\mathfrak{S}$ such that $\bar{\alpha}\bar{\beta}\bar{\alpha} = \bar{\alpha}$. Let us assume that $\bar{\alpha} \neq 0$, from

which we can deduce that $N_\alpha \notin \mathfrak{M}^*$. Consequently there exists a maximal element $N \in \mathfrak{M}$ such that $N \neq 0$, $N_\alpha \cap N = 0$, and $N \subseteq M_\alpha$. The maximality of N implies that $N_\alpha + N \in \mathfrak{M}^*$. Now α is an isomorphism of N into M , so there exists some $\beta \in \mathfrak{R}$ such that $\beta(\alpha x) = x$, $x \in N$. Evidently $(\alpha\beta\alpha)x = \alpha x$, $x \in N$; $(\alpha\beta\alpha)x = 0$, $x \in N_\alpha$, so that $\alpha\beta\alpha - \alpha \in \mathfrak{S}$. Thus $\bar{\alpha}\bar{\beta}\bar{\alpha} = \bar{\alpha}$ and the theorem follows.

The set of all submodules of M other than 0 is denoted by $\mathfrak{M} - (0)$.

COROLLARY. *The extended centralizer of R over M is a division ring if and only if $\mathfrak{M}^* = \mathfrak{M} - (0)$.*

If $N \cap N' \neq 0$ for every pair of nonzero submodules of M , then the radical \mathfrak{S} and \mathfrak{R} consists of the non-isomorphisms of \mathfrak{R} . If $\bar{\alpha}\bar{\beta} = 0$, $\bar{\alpha}, \bar{\beta} \in \mathfrak{R}/\mathfrak{S}$, then either α or β is a non-isomorphism (since the product of isomorphisms is an isomorphism), that is, either $\bar{\alpha} = 0$ or $\bar{\beta} = 0$. Thus $\mathfrak{R}/\mathfrak{S}$ is a domain of integrity, and hence a division ring.

On the other hand, if $N \cap N' = 0$ for some pair of nonzero submodules of M , no loss of generality results from assuming that $N + N' \in \mathfrak{M}^*$. Then it is possible to find elements $\alpha, \beta \in \mathfrak{R}$ with $N_\alpha = N$, $N_\beta = N'$, and $\alpha x = x$, $x \in N'$, $\beta x = x$, $x \in N$. Obviously $\bar{\alpha}, \bar{\beta} \neq 0$, while $\bar{\alpha}\bar{\beta} = 0$. This proves the corollary to Theorem 2.

In the ring R , let $I_a = \{x \mid x \in R, ax = 0\}$, the right annihilator of the element a of R . The element a of R is called (right) *singular* in case $I_a \cap I \neq 0$ for all nonzero right ideals I of R . The set S of all (right) singular elements of R is shown to be an ideal in much the same way that \mathfrak{S} is shown to be an ideal in \mathfrak{R} . We shall call S the (right) *singular ideal* of R .

If the ring R is a subring of the ring Q , Q is called a (right) *quotient ring* of R if Q has a unit element, and for every $\alpha \in Q$, $\alpha \neq 0$, there exist elements a, b in R with $b \neq 0$ such that $\alpha a = b$. If, in addition, Q is a regular ring, then Q is called a regular quotient ring of R .

Let us now show that a ring having a regular quotient ring Q has its singular ideal equal to zero. For every $\beta \in Q$, we denote by I_β the right annihilator of β in R ; thus I_β is a right ideal in R . If $a \in R$, $a \neq 0$, there exists $\alpha \in Q$ such that $a\alpha a = a$. Evidently $\epsilon = \alpha a$ and $1 - \epsilon$ are idempotents, and $I_\epsilon = I_a$. If $I_a \neq 0$, $\epsilon \neq 1$ and $1 - \epsilon \neq 0$. By assumption, there exist $c, d \in R$ with $d \neq 0$ such that $\epsilon c = d$. Since $d \in I_{1-\epsilon}$, both I_ϵ and $I_{1-\epsilon}$ are unequal to zero. Certainly $I_\epsilon \cap I_{1-\epsilon} = 0$, and therefore a is not singular in R . Note that if $a \in R$ and $I_a = 0$, then $\alpha a = 1$; that is, the elements of R having no nonzero right annihilator have left inverses in Q .

A possible choice of a right R -module for any ring R is the additive group R^+ of R . Then the left multiplications of R , that is, the

mappings a' of R^+ defined by $a'x = ax$, $x \in R^+$, $a \in R$, are in the centralizer of R over R^+ . If α is a semi-automorphism of R^+ , and $\alpha x = y$ for some $x, y \in R^+$, then it follows easily that $\alpha x' = y'$.

We now assume that the (right) singular ideal of R is zero, and that $M = R^+$. Then for any $a \in R$, $a \neq 0$, $N_a \in \mathfrak{M}$, and therefore $a' \notin \mathfrak{S}$, the radical of \mathfrak{R} . As a matter of fact, if $\alpha \in \mathfrak{R}$, $\alpha M_a \neq 0$, then $\alpha x = y \neq 0$ for some $x, y \in M$, and $\alpha x' = y' \notin \mathfrak{S}$ so that $\alpha \notin \mathfrak{S}$. It follows that the radical of \mathfrak{R} consists of all 0_N , $N \in \mathfrak{M}^*$, and that the elements of the extended centralizer Q are essentially the maximal semi-automorphisms (since $\alpha \leq \beta$ implies $\bar{\alpha} = \bar{\beta}$) of M . Thus R is (isomorphic to) a subring of Q , and in view of Theorem 2, Q is a regular quotient ring of R . We have proved the following theorem.²

THEOREM 3. *A ring R has a (right) regular quotient ring if and only if the (right) singular ideal of R is zero.*

The extended centralizer Q of R over R^+ is the universal quotient ring of R in the sense that any quotient ring P of R is a subring of Q . For if $\alpha \in P$, $\alpha \neq 0$, α can be thought of as a semi-automorphism of R^+ with $M_\alpha = I^+$ where $I = \{a \mid a \in R, \alpha a \in R\}$. Since $\alpha M_\alpha \neq 0$, P is a subring of Q .

In case R is a domain of integrity, its singular ideal is zero, and therefore R has a regular quotient ring Q . By the corollary of Theorem 2, Q is a division ring if and only if $\mathfrak{M}^* = \mathfrak{M} - (0)$, that is, if and only if $xR \cap yR \neq 0$ for every pair of nonzero elements $x, y \in R$. This yields the following corollary.

COROLLARY. *Any domain of integrity R has a (right) regular quotient ring Q , and each nonzero element of R has a left inverse in Q . The ring Q is a division ring if and only if every pair of nonzero elements of R has a common nonzero right multiple.*

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² O. Goldman, Bull. Amer. Math. Soc. vol. 52 (1946) p. 130, gives necessary and sufficient conditions that a ring be a subring of a regular ring.