ON BERNSTEIN'S APPROXIMATION PROBLEM

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1. Let $K(x)$ be an even function, defined on $(-\infty, \infty)$ and such that

$$1 \leqq K(x) \leqq \infty.$$

As to its regularity, let us suppose that $K(x)$ is lower semi-continuous. Let C_K be the class of functions $f(x)$ which are continuous on $(-\infty, \infty)$ and satisfy the condition

$$\lim_{|x|=\infty} \frac{f(x)}{K(x)} = 0, \tag{1}$$

and assume that every polynomial satisfies this condition. Bernstein's approximation problem is to find the conditions on $K(x)$ under which every function $f(x)$ in C_K can be approximated by a polynomial $P(x)$ so that

$$|f(x) - P(x)| < \epsilon K(x), \qquad -\infty < x < \infty,$$

where ϵ is a given number, $\epsilon > 0$. If the approximation is possible we say that K is of type B.

This problem has been treated by various authors, but the results are scattered, and no complete solution has been given even in the case when $\log K(x)$ is convex in $\log x$. We shall be concerned chiefly with the question of finding necessary conditions, which is of interest also for Stieltjes' moment problem. Let us finally note that analogous considerations are valid if we approximate $f(x)$ by polynomials so that

$$\int_{-\infty}^{\infty} |f(x) - P(x)|^p \frac{dx}{K(x)} < \epsilon, \qquad p \geqq 1.$$

2. A sufficient condition can be found by means of the following theorem:

THEOREM. *Let $h(z)$, $z = x + iy$, be holomorphic in $x > 0$ and suppose that*

$$h(\rho n) = 0, \quad n = 1, 2, \cdots; 0 < \rho < \infty,$$

and

Received by the editors December 6, 1950.

$$|h(x+iy)| \leq \text{const. } e^{m(x)}.$$

These conditions imply $h(z)\equiv 0$ *if and only if*

$$\int^{\infty} e^{-(\rho/2)(m^*(x)/x)}dx = \infty, \tag{2}$$

where $m^*(x)$ *is the largest convex minorant of* $m(x)$.

This theorem has a proof which is similar to that of a theorem of Fuchs,[1] making use of bilateral Laplace transforms. Since theorems of this kind are well known, we omit the proof. Let us only note that we can a priori replace $m(x)$ by $m^*(x)$.

3. Let now μ be a measure of bounded variation on $[0, \infty)$ such that

$$\int_0^{\infty} \frac{t^{2n}}{K(t)}\, d\mu(t) = 0, \qquad n = 0, 1, 2, \cdots .$$

The function

$$h(z) = \int_0^{\infty} \frac{t^z}{K(t)}\, d\mu(t)$$

satisfies the conditions in the theorem above with

$$e^{m(x)} = \sup_{t>0} \frac{t^x}{K(t)} \cdot$$

We thus conclude that $h(z)\equiv 0$, if (2) diverges with $\rho=2$. This integral diverges if and only if the integral

$$\int^{\infty} \frac{\log K^*(x)}{x^2}\, dx \tag{3}$$

diverges,[2] where $\log K^*(x)$ is the largest minorant of $\log K(x)$ that is convex in $\log x$. Thus if (3) diverges, we conclude that

$$L(f) = \int_0^{\infty} \frac{f(t)}{K(t)}\, d\mu(t) = 0$$

for every continuous function $f(x)$ on $[0, \infty)$ satisfying (1). Let us call this class of functions C_K^+. Then, according to a theorem of

[1] W. H. J. Fuchs, *On a generalization of the Stieltjes moment problem*, Bull. Amer. Math. Soc. vol. 52 (1946).

[2] A. Ostrowski, *Ueber quasianalytische Funktionen und Bestimmtheit asymptotischer Entwicklungen*, Acta Math. vol. 53 (1929).

Banach and the regularity of $K(x)$, every function f in C_K^+ can be approximated by an even polynomial, which implies that $K(x)$ is of type B.

THEOREM. *A sufficient condition for* $K(x)$ *to be of type* B *is that* (3) *diverge.*

If, on the other hand,[3]

$$\int_{-\infty}^{\infty} \frac{\log K(x)}{1 + x^2} dx < \infty, \tag{4}$$

let $\{P_n(x)\}$ be a sequence of polynomials such that

$$\lim_{n=\infty} \sup_{x} \left| \frac{f(x) - P_n(x)}{K(x)} \right| = 0$$

for a certain function $f(x)$ in C_K. Then

$$| P_n(x) | \leqq \text{const. } K(x)$$

uniformly in n. We have, for $y \neq 0$,

$$\begin{aligned} \log | P_n(x + iy) | &\leqq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{| y | \log^+ | P_n(t) | dt}{(x - t)^2 + y^2} \\ &\leqq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{| y | \log K(t) dt}{(x - t)^2 + y^2} + \text{const.} \end{aligned}$$

The family $\{P_n(z)\}$ is thus normal in the half-plane $y > 0$, say, and we can conclude that $f(x)$ is the limit function of a function $F(z)$ which is holomorphic for $y > 0$:

$$f(x) = \lim_{y=+0} F(x + iy) \quad \text{a.e.}$$

This is not true for an arbitrary function $f(x)$ in C_K. We have thus proved that $K(x)$ is not of type B, if the integral (4) converges.

THEOREM. *A necessary condition that* $K(x)$ *be of type* B *is that the integral* (4) *diverge.*

Combining the results, we find the following theorem:[4]

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[3] See T. Hall, *Sur l'approximation polynomiale des fonctions continues d'un variable reélle*, 9. Congr. des Math. scand., 1939.

[4] Compare S. Mandelbrojt, *Some theorems connected with the theory of infinitely differentiable functions*, Duke Math. J. vol. 11 (1944), and *Théorèmes d'unicité*, Ann. École Norm. (3) vol. 65 (1948).

THEOREM. *If* $\log K(x)$ *is a convex function of* $\log x$ *and* $K(0)<\infty$, *then* $K(x)$ *is of type* B *if and only if the integral* (3) *diverges.*

4. It would be very natural to suppose that the condition that (3) converges is necessary and sufficient for an arbitrary function $K(x)$.[3] This is, however, not the case, although it is true for a very large class of functions $K(x)$. We have to require a certain regularity in the relation between $K(x)$ and its minorant $K^*(x)$.

We introduce the concept of a *supporting set.* Let $\{\lambda_n\}_0^\infty$ be a sequence of real numbers,

$$1=\lambda_0<\lambda_1<\lambda_2<\cdots,\quad \lambda_n-\lambda_{n-1}\geqq\delta>0.$$

Then we call $\{\lambda_n\}_0^\infty$ a supporting set if the conditions

$$K(\lambda_n)\leqq MK^*(\lambda_n),\qquad n=0,1,2,\cdots,$$

together with the convergence of (3) imply that K is not of type B, M being a certain constant.

We have the following criterion on supporting sets.

THEOREM. $\{\lambda_n\}_0^\infty$ *is a supporting set if and only if the polynomials* $P(x)$ *satisfying the inequalities*

$$|P(\pm\lambda_n)|\leqq K^*(\lambda_n),\qquad n=0,1,2,\cdots, \tag{5}$$

where the integral (3) *converges, are uniformly bounded at the origin*:

$$|P(0)|\leqq M_{K^*}<\infty.$$

The condition is evidently sufficient. To prove the necessity, suppose that $P(0)$ can be chosen arbitrarily large and consider the function

$$K(x)=\begin{cases}K^*(x), & x=\lambda_n,\ n=0,1,2,\cdots,\\ +\infty, & \text{everywhere else.}\end{cases}$$

Suppose that

$$L(x^{2n})=\sum_{\nu=0}^{\infty}\frac{x_\nu}{K(\lambda_\nu)}\lambda_\nu^{2n}=0,\qquad n=0,1,2,\cdots, \tag{6}$$

where

$$\sum_0^\infty|x_\nu|<\infty.$$

Let $P(x)$ be an even polynomial satisfying the inequalities (5). Then

$$\left|P(0)\sum_{\nu=0}^{\infty}\frac{x_\nu}{K(\lambda_\nu)\lambda_\nu^2}\right| \leqq \sum_{\nu=0}^{\infty}\frac{|x_\nu|\,|P(\lambda_\nu)|}{\lambda_\nu^2 K(\lambda_\nu)} \leqq \sum_{\nu=0}^{\infty}|x_\nu|.$$

Since $P(0)$ can be chosen arbitrarily large, we conclude that

$$\sum_{\nu=0}^{\infty}\frac{x_\nu}{K(\lambda_\nu)\lambda_\nu^2} = 0.$$

It follows by induction that the function

$$h(z) = \sum_{\nu=0}^{\infty}\frac{x_\nu}{K(\lambda_\nu)}\lambda_\nu^{-2z},$$

which is holomorphic and bounded in the right half-plane, vanishes for $z=1, 2, \cdots$. This yields $h(z)\equiv 0$, which implies

$$x_0 = x_1 = \cdots = x_\nu = \cdots = 0$$

and thus

$$L(f) = \sum_{\nu=0}^{\infty}\frac{x_\nu}{K(\lambda_\nu)}f(\lambda_\nu) \equiv 0, \qquad f \in \overset{+}{C}_K.$$

We conclude that the particular function $K(x)$ chosen above is of type B, which proves the assertion.

A set must be sufficiently dense, if it is to be a supporting set. This follows from the following theorem.

THEOREM. *A necessary condition that $\{\lambda_n\}_0^\infty$ be a supporting set is*

$$\limsup_{n=\infty}\frac{\lambda_n}{n} < \infty.$$

From this theorem and the definition of supporting sets it follows that there exist functions $K(x)$ of type B for which the integral (3) converges.

Suppose that the upper limit is infinite. Let $\{n_\nu\}_1^\infty$ be a sequence of integers to be determined later and set

$$K^*(x) = \sup_\nu\left[\frac{|x|^{n_\nu}}{\lambda_1\cdot\lambda_2\cdot\,\cdots\,\cdot\lambda_{n_\nu}}\right]^2 = \sup_\nu\frac{|x|^{2n_\nu}}{e^{2\Lambda_\nu}}.$$

Then $\log K^*(x)$ is convex in $\log x$. If

$$k^*(t) = \log K^*(e^t),$$

we have

$$\int^{\infty} \frac{\log K^*(x)}{x^2} dx = \int^{\infty} k^*(t)e^{-t}dt.$$

Now

$$\int^{\infty} k^*(t)e^{-t}dt \leqq \sum^{\infty} \int_{\Lambda_\nu/n_\nu}^{\infty} (2n_\nu t - 2\Lambda_\nu)e^{-t}dt$$
$$= 2\sum^{\infty} n_\nu e^{-\Lambda_\nu/n_\nu}.$$

Furthermore, by Stirling's formula,

$$\Lambda_\nu = \sum_{k=1}^{n_\nu} \log \lambda_k \geqq n_\nu \log n_\nu + A_\nu n_\nu,$$

where, according to our hypothesis, $A_\nu \to \infty$ for an appropriate sequence $\{n_\nu\}_1^\infty$. We choose $\{n_\nu\}_1^\infty$ so that

$$\sum^{\infty} e^{-A_\nu} < \infty$$

and find

$$\int^{\infty} \frac{\log K^*(x)}{x^2} dx \leqq 2\sum^{\infty} e^{-A_\nu} < \infty.$$

On the other hand the polynomials

$$P_\nu(x) = \prod_0^{n_\nu}\left(1 - \frac{x^2}{\lambda_k^2}\right)$$

satisfy the inequalities (5) for our choice of $K^*(x)$. From this, it follows by the preceding theorem[5] that $\{\lambda_n\}_0^\infty$ is not a supporting set, which proves the theorem.

If the sequence $\{\lambda_n\}_0^\infty$ satisfies the condition (5) and has some additional arithmetical regularity, then it is a supporting set. We shall content ourselves with the following result, showing that the class of functions $K(x)$ for which the divergence of (3) is a necessary and sufficient condition that $K(x)$ be of type B, is very large.

THEOREM. *If* $0<c<\infty$ *and*

$$\lambda_n = cn + O(n^\alpha), \qquad \alpha < 1,$$

then $\{\lambda_n\}_0^\infty$ *is a supporting set.*

[5] See T. Hall, *On polynomials bounded at an infinity of points*, Uppsala, 1950, p. 42 ff.

Let us sketch the proof. The integral function

$$F(z) = \prod_0^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

has the following properties:[6] for $|x - \lambda_n| \geqq \delta/3$

$$\log |F(x+iy)| \geqq \log F(iy) - Ax \tag{7}$$
$$\geqq 2\eta |y| - Ax, \qquad \eta > 0, A < \infty,$$

and

$$\log |F'(\lambda_n)| \geqq -B\lambda_n^\alpha \log \lambda_n, \qquad B < \infty.$$

Let $K^*(x)$ be a given minorant function such that (3) converges. If for $x \geqq 1$

$$\log G(x) = -\log K^*(x) - Bx^\alpha \log x - 3 \log x,$$

then

$$\int^\infty \frac{\log G(x)}{x^2} dx > -\infty,$$

and there exists an even integral function $H(z) \not\equiv 0$ so that

$$|H(x+iy)| \leqq e^{\eta|y|} \tag{8}$$

and $|H(x)| \leqq G(x)$. The function

$$\phi(\sigma + i\tau) = \phi(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-tz} \frac{H(z)}{F(z)} dz$$

is holomorphic for $|\tau| < \eta$ by (7) and (8). Furthermore $\phi^{(2k+1)}(0) = 0$, $k = 0, 1, 2, \cdots$, since H and F are even.

If now $\sigma > A$ and we move the path of integration to the right, we get by the zeros of F a representation

$$\phi(t) = \sum_{\nu=0}^\infty \frac{x_\nu}{K^*(\lambda_\nu)\lambda_\nu} e^{-\lambda_\nu t}, \qquad \sum_0^\infty |x_\nu| < \infty,$$

for $\sigma > A$, which by analytic continuation holds for $\sigma \geqq 0$. Since $\phi^{(2k+1)}(0) = 0$, we find

$$\sum_{\nu=0}^\infty \frac{x_\nu}{K^*(\lambda_\nu)} \lambda_\nu^{2n} = 0, \qquad n = 0, 1, 2, \cdots.$$

[6] For estimations of this kind see N. Levinson, *Gap and density theorems*, Amer. Math. Soc. Colloquium Publications, vol. 26, 1940.

For every $K(x)$ such that

$$K(\lambda_\nu) \leqq MK^*(\lambda_\nu), \quad L(f) = \sum_{\nu=0}^{\infty} \frac{x_\nu}{K^*(\lambda_\nu)} f(\lambda_\nu), \qquad f \in C_K^+,$$

vanishes for $f(x) = x^{2n}$, $n = 0, 1, 2, \cdots$, without vanishing identically. The set $\{\lambda_n\}_0^\infty$ is thus a supporting set, which was our assertion.

5. To illustrate the connection between this approximation problem and the theory of quasianalytic functions, let us prove Carleman's summation theorem for the Taylor series of a function belonging to a quasianalytic class on $(-\infty, \infty)$. This proof gives an interesting interpretation of the summation matrix.

Let $C\{m_n\}$ be a quasianalytic class of functions on $(-\infty, \infty)$. We may assume that the sequence $\{\log m_n\}$ is convex. If

$$\mu_n = \max (m_{n+2}, n!),$$

then $C\{\mu_n\}$ is quasianalytic, that is,

$$\sum_{n=1}^{\infty} \frac{1}{(\mu_n)^{1/n}} = \infty,$$

since the corresponding series for $\{m_n\}_1^\infty$ diverges. Thus if

$$\mu(t) = \sup_{n \geqq 1} \frac{|t|^n}{\mu_n},$$

then

$$\int_{-\infty}^{\infty} \frac{\log \mu(t)}{1 + t^2} dt = \infty,$$

and we conclude that $\mu(kt)$ is of type B for all positive constants k.

Suppose now that $f(x)$ belongs to $C\{m_n\}$, and introduce the auxiliary function

$$h(x) = x^{-2}(f(x) - f(0) - f'(0) \sin x).$$

Then $h(x)$ belongs to $C\{\mu_n\}$ and for some constant a we have[7]

$$(9) \qquad |h^{(n)}(x)| \leqq a^n \mu_n \cdot \frac{1}{1 + x^2}, \qquad n = 0, 1, 2, \cdots.$$

If now

$$H(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-ixt} h(x) dx,$$

[7] See T. Bang, *Om quasianalytiske Funktioner*, Copenhagen, 1946, pp. 21 and 34.

partial integrations together with (9) give

$$| H(t) | \leqq \text{const. } \inf_n \frac{a^n \mu_n}{| t |^n} = \frac{\text{const.}}{\mu(kt)}.$$

Thus the inversion formula

$$h(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{ixt} H(t) dt$$

holds. Let $\{C_\nu\}_0^n$ be an arbitrary sequence and form

$$\begin{aligned} h(x) - \sum_{\nu=0}^{n} \frac{h^{(\nu)}(0)}{\nu!} x^\nu C_\nu \\ = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \left\{ e^{ixt} - \sum_{\nu=0}^{n} \frac{(ixt)^\nu}{\nu!} C_\nu \right\} H(t) dt. \end{aligned} \tag{10}$$

If now $\{k_m\}_1^\infty$ and $\{\epsilon_m\}_1^\infty$ are two sequences of positive numbers tending to zero and $\{R_m\}_1^\infty$ is a sequence tending to infinity, then to every m there corresponds an integer n and a sequence $\{C_{\nu m}\}_0^n$ such that for $|x| < R_m$

$$\left| e^{ixt} - \sum_{\nu=0}^{n} \frac{(ixt)^\nu}{\nu!} C_{\nu m} \right| < \epsilon_m \mu(k_m t) \frac{1}{1+t^2}, \qquad -\infty < t < \infty.$$

For this sequence, we have by (10)

$$\left| h(x) - \sum_{\nu=0}^{n} \frac{h^{(\nu)}(0)}{\nu!} x^\nu C_{\nu m} \right| \to 0, \qquad m \to \infty,$$

uniformly for $|x| < R$.

Now, returning to our original function $f(x)$, we see that its Taylor series is summable to $f(x)$ by the summation method defined by the matrix

$$\begin{Vmatrix} 1 & 1 & C_{01} & C_{11} \cdots 0 & 0 \cdots \\ 1 & 1 & C_{02} & C_{12} \cdots \cdots \\ 1 & 1 & \cdots \cdots & \\ \cdots & & & \end{Vmatrix},$$

which depends on the function $\mu(t)$ only.

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