THE FIRST VARIATION OF AN INDEFINITE WIENER INTEGRAL

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1. **Introduction.** It is the purpose of this paper to obtain a formula for δG when G is the indefinite Wiener integral

$$G(u) = \int_{x(t)\leqq u(t)}^{W} F(x)d_W x. \tag{1}$$

Here x is understood to be a variable point in the Wiener space C of continuous functions $x(t)$ defined on $0\leqq t\leqq 1$ and vanishing at $t=0$. The integration is performed over the set $\mathfrak{S}_u$ of elements $x(t)$ of C which satisfy for all t the inequality

$$x(t) \leqq u(t).$$

The function $u(t)$ need not be a member of C, but can be any Borel measurable function defined on $0\leqq t\leqq 1$, and may even be permitted to take on infinite values.

The Wiener integral of a functional is simply the Lebesgue integral of the functional with respect to Wiener's measure [4][1] in C. This measure is not invariant under translations, but is in other respects a Lebesgue measure based on intervals of the form

$$I: \qquad \alpha_j < x(t_j) < \beta_j \qquad \text{(where } 0 < t_1 < t_2 < \cdots < t_n \leqq 1),$$

having the measure

$$m_W(I) = \frac{1}{(\pi^n t_1(t_2-t_1)\cdots(t_n-t_{n-1}))^{1/2}}\int_{\alpha_n}^{\beta_n}\cdots\int_{\alpha_1}^{\beta_1}\exp\left\{-\frac{\zeta_1^2}{t_1} - \frac{(\zeta_2-\zeta_1)^2}{t_2-t_1} - \cdots - \frac{(\zeta_n-\zeta_{n-1})^2}{t_n-t_{n-1}}\right\}d\zeta_1\cdots d\zeta_n.$$

We denote the Wiener integral of a measurable functional $F(x)$ over a measurable set $S\subset C$ by

$$\int_S^W F(x)d_W x.$$

In case S is not contained in C but SC is measurable, we define

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$m_W(S) = m_W(SC) \text{ and } \int_S^W F(x)d_Wx = \int_{SC}^W F(x)d_Wx.$$

In addition to finding δG when G is given by (1), we shall also find certain transformation formulas for Wiener integrals taken over the whole of C. In particular, we shall obtain what may be considered as a formula for integration by parts in function space.

Finally, we show that these formulas may be used to evaluate certain Wiener integrals. As an example we show that

$$\int_C^W \left[\log \int_s^1 \alpha(t)e^{x(t)}dt\right] x(s)d_Wx = \frac{s}{2} \tag{2}$$

when $0<s<1$, if $\alpha(t)$ is non-negative and of class L_1 and not equivalent to zero on $s \leqq t \leqq 1$.

2. Sub-summable functionals.

Definitions. *Let $f(u)$ be a real or complex function defined on a set S of an abstract space in which a measure is defined. Then $f(u)$ will be called "sub-summable" on S if there exists a function $g(u)$ which is summable on a measurable set $\mathfrak{S}$ containing S and satisfies on S the inequality $|f(u)| \leqq g(u)$.*

It is clear that if $f(u)$ is also measurable on S, then it is summable on S (and, of course, S is also measurable).

Lemma. *Let $y_0(t) \in C$ be absolutely continuous and have a derivative $y_0'(t)$ which is essentially of bounded variation[2] on $[0, 1]$; let $\mathfrak{S}$ be a Wiener measurable subset of C; and for each positive λ let $\mathfrak{S}^\lambda$ be the set of all $x(t)$ of the form $u(t)+hy_0(t)$, where $u \in \mathfrak{S}$ and $|h| \leqq \lambda$. Let $\epsilon>0$, $\eta>0$ and let $F(x)$ be a functional defined on $\mathfrak{S}^{\epsilon+\eta}$ such that*

$$\sup_{|h| \leqq \eta} |F(x + hy_0)| \tag{3}$$

is sub-summable on $\mathfrak{S}^\epsilon$. Then if $P(\omega)$ is any polynomial, it follows that there exists $\eta_1>0$ for which

$$\sup_{|h| \leqq \eta_1} |F(x + hy_0)| \cdot \exp\left\{2\eta_1 \left| \int_0^1 y_0'(t)dx(t)\right|\right\} P\left(\int_0^1 y_0'(t)dx(t)\right) \tag{4}$$

is sub-summable on $\mathfrak{S}$.

[2] Here and elsewhere in this paper the requirement that a function be "essentially of bounded variation" can be replaced by the requirement that it be "of class L_2" if Stieltjes integrals are interpreted as Paley-Wiener-Zygmund integrals [3, 1].

For the proof, choose $\eta_2>0$ so that $\eta_2<\epsilon$ and $\eta_2<\eta/2$, and apply the translation theorem [2; 1] to a functional $G(x)$ which is summable on a measurable set containing $\mathfrak{S}^\epsilon$ and which satisfies on $\mathfrak{S}^\epsilon$

$$G(x) \geqq \sup_{|h|\leqq\eta} |F(x+hy_0)|. \tag{5}$$

We translate by $\eta_2 y_0$ and also by $-\eta_2 y_0$, and obtain

$$\begin{aligned}\int^W_{(x\mp\eta_2 y_0)\in\mathfrak{S}} G(x)d_W x &= \exp\left\{-\eta_2^2\int_0^1 [y_0'(t)]^2 dx(t)\right\}\\ &\cdot \int^W_{\mathfrak{S}} G(x\pm\eta_2 y_0)\exp\left\{\mp 2\eta_2\int_0^1 y_0'(t)dx(t)\right\}d_W x.\end{aligned} \tag{6}$$

Here the existence of the first member follows from the fact that we are integrating over a measurable subset of $\mathfrak{S}^\epsilon$, and the existence of the second follows from that of the first by the translation theorem. Moreover, since $2\eta_2<\eta$, we have by (5) for $x\in\mathfrak{S}$ and for both upper and lower signs,

$$\sup_{|h|\leqq\eta_2} |F(x+hy_0)| \leqq G(x\pm\eta_2 y_0),$$

and hence it follows from the existence of the Wiener integral in the second member of (6) that

$$\sup_{|h|\leqq\eta_2} |F(x+hy_0)| \exp\left\{\pm 2\eta_2\int_0^1 y_0'(t)dx(t)\right\}$$

is sub-summable on $\mathfrak{S}$. Thus the maximum of these two functionals is also sub-summable; that is,

$$\sup_{|h|\leqq\eta_2} |F(x+hy_0)| \exp\left\{2\eta_2\left|\int_0^1 y_0'(t)dx(t)\right|\right\}$$

is sub-summable on $\mathfrak{S}$. The sub-summability of (4) on $\mathfrak{S}$ follows immediately for positive $\eta_1<\eta_2$, and hence the lemma is established.

3. The first variation of a Wiener integral and vice versa.

THEOREM I. *Let $y_0(t)\in C$ be absolutely continuous and have a derivative $y_0'(t)$ which is essentially of bounded variation*[2] *on $0\leqq t\leqq 1$, and let $F(x)$ be a Wiener summable functional over the set $\mathfrak{S}_{u_0+2\epsilon}$, where $u_0(t)$ is Borel measurable on $0\leqq t\leqq 1$ (and may even be infinite there), $\epsilon>0$, and*

$\mathfrak{S}_u$: $\qquad x(t)\leqq u(t), \qquad 0\leqq t\leqq 1;\ x\in C.$

Let $F(x)$ have a first variation

$$\delta F \equiv \delta F(x \mid y_0) = \frac{d}{dh} F(x + hy_0)\Big]_{h=0} \tag{7}$$

for all $x \in \mathfrak{S}_{u_0+2\epsilon}$. *Then if* $0 < \eta \max_{0 \leqq t \leqq 1} |y_0(t)| \leqq \epsilon$ *and*

$$\sup_{|h| \leqq \eta} |\delta F(x + hy_0 \mid y_0)| \tag{8}$$

is Wiener sub-summable in x *on* $S_{u_0+\epsilon}$, *it follows that the functional*

$$G(u) = \int^{W}_{x(t) \leqq u(t)} F(x) d_W x \tag{9}$$

has a first variation

$$\delta G = \delta G(u \mid y_0) = \frac{d}{dh} G(u + hy_0)\Big]_{h=0} \tag{10}$$

whenever $u(t) \leqq u_0(t)$ *on* $0 \leqq t \leqq 1$ *and* u *is Borel measurable. Moreover the value of the variation is given by the following integrals* (*which necessarily exist*):

$$\begin{aligned} \delta G(u \mid y_0) &= \int^{W}_{x(t) \leqq u(t)} \delta F(x \mid y_0) d_W x \\ &\quad - 2 \int^{W}_{x(t) \leqq u(t)} F(x) \left[\int_0^1 y_0'(t) dx(t) \right] d_W x. \end{aligned} \tag{11}$$

For the proof, we note that if $x + hy_0 \in \mathfrak{S}_{u_0+2\epsilon}$,

$$\begin{aligned} \delta F(x + hy_0 \mid y_0) &= \frac{d}{d\lambda} F(x + hy_0 + \lambda y_0)\Big]_{\lambda=0} \\ &= \frac{d}{d\mu} F(x + \mu y_0)\Big]_{\mu=h} = \frac{d}{dh} F(x + hy_0); \end{aligned}$$

and since the first member of this equation exists, so does the last.

Again, it is clear that $\mathfrak{S}_{u+2\epsilon}$ is convex, so that if $x \in \mathfrak{S}_{u+2\epsilon}$ and $x + hy_0 \in \mathfrak{S}_{u+2\epsilon}$, we have $x + \theta hy_0 \in \mathfrak{S}_{u+2\epsilon}$ for all θ in $0 \leqq \theta \leqq 1$. Thus by the law of the mean we obtain $F(x+hy_0) = F(x) + h\delta F(x + \theta hy_0 \mid y_0)$ for some θ in $0 < \theta < 1$ depending on h. Hence it follows from the sub-summability of (8) and of $F(x)$ that

$$\sup_{|h| \leqq \eta} |F(x + hy_0)| \tag{12}$$

is sub-summable on $\mathfrak{S}_{u_0+\epsilon}$.

Now for $|h| \leqq \eta$ and u a Borel measurable function satisfying $u(t) \leqq u_0(t)$, we have by the translation theorem (which guarantees the existence of the last member)

$$\begin{aligned} G(u + hy_0) &= \int^W_{x(t)-hy_0(t)\in \mathfrak{S}_u} F(x)d_W x \\ &= \exp\left\{- h^2 \int_0^1 [y_0'(t)]^2 dt\right\} \int^W_{\mathfrak{S}_u} F(x + hy_0) \\ &\qquad\qquad \cdot \exp\left[- 2h \int_0^1 y_0'(t)dx(t)\right] d_W x. \end{aligned}$$

Differentiating formally with respect to h and then setting $h=0$, we obtain

$$\begin{aligned} \delta G(u \mid y_0) &= \frac{d}{dh} G(u + hy_0)\Big]_{h=0} \\ &= \int^W_{\mathfrak{S}_u}\left[\frac{d}{dh}\left\{F(x + hy_0)\right.\right. \\ &\qquad\qquad \left.\left.\cdot \exp\left[- 2h \int_0^1 y_0'(t)dx(t)\right]\right\}\right]_{h=0} d_W x \\ &= \int^W_{\mathfrak{S}_u} \delta F(x \mid y_0) d_W x \\ &\quad - 2\int^W_{\mathfrak{S}_u} F(x)\left[\int_0^1 y_0'(t)dx(t)\right] d_W x. \end{aligned} \tag{13}$$

To justify this differentiation under the integral sign (and incidentally show that all members of (13) exist), we must show that the differentiated integrand is dominated for small h by a summable functional; that is, we must show that

$$\begin{aligned} \sup_{|h|\leqq \eta_1} \Bigg| &\left\{\delta F(x + hy_0 \mid y_0) - 2F(x + hy_0) \int_0^1 y_0'(t)dx(t)\right\} \\ &\qquad \cdot \exp\left\{- 2h \int_0^1 y_0'(t)dx(t)\right\}\Bigg| \end{aligned} \tag{14}$$

is sub-summable on $\mathfrak{S}_u$ for some $\eta_1 > 0$. But it follows from the sub-summability of (8) on $\mathfrak{S}_{u_0+\epsilon}$ and the lemma that for some $\eta_2 > 0$

$$\sup_{|h|\leqq \eta_2} | \delta F(x + hy_0 \mid y_0) | \exp\left\{2\eta_2 \left|\int_0^1 y_0'(t)dx(t)\right|\right\}$$

is sub-summable on $\mathfrak{S}_{u_0}$. Similarly it follows from the sub-summability of (12) on $\mathfrak{S}_{u_0+\epsilon}$ and the lemma that for some $\eta_3>0$,

$$\sup_{|h|\leqq\eta_3} |F(x+hy_0)| \exp\left\{2\eta_3\left|\int_0^1 y_0'(t)dx(t)\right|\right\}\left|\int_0^1 y_0'(t)dx(t)\right|$$

is sub-summable on $\mathfrak{S}_{u_0}$. Taking $\eta_1=\min(\eta_2,\eta_3)$, we obtain the sub-summability of (14) on $\mathfrak{S}_u\subset\mathfrak{S}_{u_0}$ and hence the justification of (13), including the existence of all its members. Thus the theorem is established.

An important special case of Theorem 1 is obtained if $u(t)\equiv u_0(t)\equiv+\infty$, so that we integrate over the whole space C. In this case also $u(t)+hy_0(t)\equiv+\infty$ and $G(u+hy_0)$ is constant and $\delta G(u|y_0)=0$. We state the result as a separate theorem.

THEOREM II. *Let $y_0(t)$ be absolutely continuous and have a derivative $y_0'(t)$ which is essentially of bounded variation*[2] *on* $0\leqq t\leqq 1$, *and let $F(x)$ be a Wiener summable functional over C. Let $F(x)$ have a first variation $\delta F=\delta F(x|y_0)$ for all $x\in C$ such that*

$$\sup_{|h|\leqq\eta} |\delta F(x+hy_0|y_0)|$$

is Wiener summable in x on C for some $\eta>0$. Then it follows that

$$\int_C^W \delta F(x|y_0)d_Wx = 2\int_C^W F(x)\left[\int_0^1 y_0'(t)dx(t)\right]d_Wx. \tag{15}$$

As a corollary to Theorem II we obtain a formula for "integration by parts in function space." We replace $F(x)$ by $F(x)G(x)$.

COROLLARY. *Let $y_0(t)$ be absolutely continuous and have a derivative $y_0'(t)$ which is essentially of bounded variation*[2] *on* $0\leqq t\leqq 1$, *and let $F(x)$ and $G(x)$ be Wiener measurable functionals on C such that $F(x)G(x)$ is Wiener summable on C. Let F and G have first variations δF and δG such that*

$$G(x)\sup_{|h|\leqq\eta} |\delta F(x+hy_0|y_0)| \quad \textit{and} \quad F(x)\sup_{|h|\leqq\eta} |\delta G(x+hy_0|y_0)|$$

are Wiener summable in x on C for some $\eta>0$. Then it follows that

$$\int_C^W F(x)\delta G(x|y_0)d_Wx$$
$$= \int_C^W G(x)\left[2F(x)\int_0^1 y_0'(t)dx(t) - \delta F(x|y_0)\right]d_Wx.$$

4. The Wiener integral of a Volterra derivative.

THEOREM III. *Let $F(x)$ be a Wiener summable functional such that $F(x)\max_{0\leqq t\leqq 1}|x(t)|$ is also Wiener summable, and such that the first variation*

$$(16)\qquad \delta F = \delta F(x\mid y) = \frac{d}{dh}F(x+hy)\Big]_{h=0}$$

exists for all x and y in C and is expressible in the form

$$(17)\qquad \delta F(x\mid y) = \int_0^1 F'(x\mid t)y(t)dt,$$

where $F'(x|t)$ is measurable in the product space of x and t as well as summable in t for each x. (It is clear that $F'(x|t)$ is the Volterra derivative of $F(x)$ at each point (x, t) of the product space for which $F'(x|t)$ is continuous in (x, t).) Assume also that for each $y(t)\in C$ there exists a corresponding number $\eta=\eta(y)>0$ such that

$$(18)\qquad \sup_{|h|\leqq\eta,0\leqq t\leqq 1}|F'(x+hy\mid t)|$$

is Wiener summable in x on C. Then it follows that $\int_C^W F(x)x(t)d_Wx$ has an absolutely continuous derivative with respect to t for $0\leqq t\leqq 1$, and this derivative vanishes at $t=1$. Moreover

$$(19)\qquad \int_C^W F'(x\mid t)d_Wx = -2\frac{d^2}{dt^2}\int_C^W F(x)x(t)d_Wx$$

for almost all t on $0\leqq t\leqq 1$, and, in particular, for each t for which the left member is continuous. Specifically, (19) holds for each t for which $F'(x|t)$ is continuous in t for almost all x in C.

We shall prove that this theorem holds even when we weaken the hypotheses (16), (17), (18) by assuming that they hold not for all y in C, but only for a sequence of values of y, namely $y=y_n$ $(n=1, 2, \cdots)$, where each $y_n(t)$ has an absolutely continuous derivative $y_n'(t)$ and satisfies $y_n(0)=y_n'(1)=0$, and where the set of second derivatives $\{y_n''(t)\}$ is closed in L_2 on $0\leqq t\leqq 1$. Then if $\|y\| =\max_{0\leqq t\leqq 1}|y(t)|$, we have by (17) for each $n=1, 2, \cdots$ the inequality

$$(20)\qquad \sup_{|h|\leqq\eta_n}|\delta F(x+hy_n\mid y_n)| \leqq \|y_n\|\cdot\sup_{|h|\leqq\eta_n,0\leqq t\leqq 1}|F'(x+hy_n\mid t)|,$$

where η_n denotes $\eta(y_n)$. Thus for each n, (18) and (20) imply that the hypotheses of Theorem II hold with y_n replacing y_0, and we have from (15),

$$\int_C^W \delta F(x \mid y_n) d_W x = 2 \int_C^W F(x) \left[\int_0^1 y_n'(t) dx(t) \right] d_W x.$$

Integrating by parts in the right member, remembering that $x(0) = y_n'(1) = 0$, and using (17) in the left member, we obtain

$$\int_C^W \left[\int_0^1 F'(x \mid t) y_n(t) dt \right] d_W x = -2 \int_C^W F(x) \left[\int_0^1 x(t) y_n''(t) dt \right] d_W x.$$

Since by hypothesis $F(x) \cdot \|x\|$ is summable, we may apply the Fubini theorem to the right member, and since (18) is summable we may apply it to the left member. Thus

$$\int_0^1 y_n(t) \Psi(t) dt = -2 \int_0^1 y_n''(t) dt \int_C^W F(x) x(t) d_W x, \tag{21}$$

where

$$\Psi(t) = \int_C^W F'(x \mid t) d_W x. \tag{22}$$

We next integrate the left member of (21) by parts twice, and to simplify the notation we introduce the function

$$\phi(t) = \int_0^t du \int_1^u ds \int_C^W F'(x \mid s) d_W x$$

which obviously satisfies the conditions

$$\begin{aligned} &\phi(t) \text{ and } \phi'(t) \text{ absolutely continuous on } 0 \leqq t \leqq 1, \\ &\phi''(t) = \Psi(t) \text{ almost everywhere on } 0 \leqq t \leqq 1, \\ &\phi(0) = \phi'(1) = 0. \end{aligned} \tag{23}$$

Thus we obtain from (21) by two integrations by parts, using $y_n(0) = y_n'(1) = 0$ and (23),

$$\int_0^1 y_n''(t) \left[\phi(t) + 2 \int_C^W F(x) x(t) d_W x \right] dt = 0. \tag{24}$$

But the $y_n''(t)$ are closed (and hence complete) in $L_2(0, 1)$, and therefore (24) implies that for almost all t on $0 \leqq t \leqq 1$

$$\phi(t) = -2\int_C^W F(x)x(t)d_W x. \tag{25}$$

Actually, (25) is true for all t on the unit interval, since both sides are continuous. The continuity of the right member follows from the continuity of $x(t)$ and the summability of $F(x)\|x\|$. Differentiating (25), we obtain for all t on $0 \leqq t \leqq 1$,

$$\phi'(t) = -2\frac{d}{dt}\int_C^W F(x)x(t)d_W x.$$

From this and (23) it is clear that $\int_C^W F(x)x(t)d_W x$ has an absolutely continuous derivative on $0 \leqq t \leqq 1$ which vanishes at $t=1$. Another differentiation gives (19) for almost all t, and in particular whenever the left member is continuous. This must occur in view of (18) whenever $F'(x|t)$ is continuous in t for almost all x, and hence the theorem is established.

COROLLARY. *Theorem* III *holds when hypotheses* (16), (17), (18) *are assumed to hold only for a sequence of* $y(t)$, $\{y_n(t)\}$, *such that each* y_n *has an absolutely continuous derivative and* $y_n(0) = y_n'(1) = 0$ *and the second derivatives* $\{y_n''(t)\}$ *are closed in* $L_2(0, 1)$.

EXAMPLE. We conclude this paper by giving an example to show how Theorem II can be used to evaluate new Wiener integrals.

As our example, let

$$F(x) = \log\left[\int_0^1 \alpha(t)\exp\left\{\frac{x(t)}{\beta(t)}\right\}dt\right],$$

where the integrand is understood to vanish when $\alpha(t)$ vanishes whether the exponential exists or not, and where $\alpha(t)$ and $\beta(t)$ satisfy the following conditions. We assume $\alpha(t) \in L_1$, $\alpha(t) \geqq 0$ on $0 \leqq t \leqq 1$, $\alpha(t) > 0$ on a set of positive measure; $\beta(t) \in C$ and is absolutely continuous with a derivative essentially of bounded variation;[2] finally, we assume

$$\int_0^1 \alpha(t)\exp\left\{\frac{t}{4[\beta(t)]^2}\right\}dt < \infty;$$

where the integrand is (as above) interpreted to vanish when $\alpha(t)$ vanishes.

We first note that $F(x) \in L_p(C)$ for all positive p. For if $r > 0$,

$$|\log r|^p < p^p \max(r, r^{-1})$$

and hence

$$|F(x)|^p < p^p \max\left[\int_0^1 \alpha(t) \exp\left|\frac{x(t)}{\beta(t)}\right| dt,\right.$$
$$\left.\left\{\int_0^1 \alpha(t) \exp\left[-\left|\frac{x(t)}{\beta(t)}\right|\right]dt\right\}^{-1}\right].$$

Moreover by the Schwartz inequality

$$\int_0^1 \alpha(t) \exp\left\{-\left|\frac{x(t)}{\beta(t)}\right|\right\} dt\cdot \int_0^1 \alpha(t) \exp\left|\frac{x(t)}{\beta(t)}\right| dt \geqq \left[\int_0^1 \alpha(t)dt\right]^2,$$

so that

$$|F(x)|^p < K\int_0^1 \alpha(t) \exp\left|\frac{x(t)}{\beta(t)}\right| dt$$

where

$$K = p^p \max\left[1, \left(\int_0^1 \alpha(t)dt\right)^{-2}\right].$$

Thus

$$\begin{aligned}\int_C^W |F(x)|^p d_W x &< K\int_0^1 \alpha(t)dt \int_C^W \exp\left|\frac{x(t)}{\beta(t)}\right| d_W x \\ &= \frac{K}{\pi^{1/2}}\int_0^1 \frac{\alpha(t)dt}{t^{1/2}}\int_{-\infty}^{\infty} \exp\left[\left|\frac{s}{\beta(t)}\right| - \frac{s^2}{t}\right] ds \\ &< \frac{2K}{\pi^{1/2}}\int_0^1 \alpha(t)dt \int_{-\infty}^{\infty} \cosh\left(\frac{ut^{1/2}}{\beta(t)}\right) e^{-u^2} du \\ &= 2K\int_0^1 \alpha(t) \exp\left\{\frac{t}{4[\beta(t)]^2}\right\} dt \\ &< \infty.\end{aligned}$$

Now let $\mathcal{N}$ be the null set where $F(x)$ fails to exist, and let us define $F(x)$ to be zero on $\mathcal{N}$. We then have

$$F(x + h\beta) = \begin{cases} F(x) + h & \text{when } x \in C - \mathcal{N}, \\ 0 & \text{when } x \in \mathcal{N}, \end{cases}$$

$$\delta F(x \mid \beta) = \begin{cases} 1 & \text{when } x \in C - \mathcal{N}, \\ 0 & \text{when } x \in \mathcal{N}. \end{cases}$$

Thus the hypotheses of Theorem II (with $y_0=\beta$) are satisfied and we obtain from (15)

$$\int_C^W \left\{\log\left[\int_0^1 \alpha(t) \exp\left(\frac{x(t)}{\beta(t)}\right) dt\right]\right\}\left[\int_0^1 \beta'(t)dx(t)\right] d_W x = \frac{1}{2}\cdot$$

In particular, if $\alpha(t)=0$ when $t<s$ and $\beta(t)=s^{-1}\min(s, t)$ for some fixed s on $0<s<1$, we obtain formula (2) given in the introduction.

As another special case, take $\beta(t)=t/2$ and $\alpha(t)=\phi(t)\exp(-t^{-1})$, where $\phi(t)$ is non-negative and summable and not equivalent to zero on $0\leqq t\leqq 1$. Clearly the required conditions on α and β are satisfied, and we have

$$\int_C^W x(1)\left[\log \int_0^1 \phi(t) \exp\left[\frac{2x(t)-1}{t}\right] dt\right] d_W x = 1.$$

Other interesting formulas can be obtained by using the formula for "integrating by parts in function space."

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