AN EMBEDDING OF PI-RINGS

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1. **Introduction.** It is well known that a commutative ring which has no nonzero nilpotent ideals is isomorphic to a subring of a complete direct sum of commutative fields (McCoy [1]).[1] In this note, this fact is generalised to rings which satisfy a polynomial identity (PI-rings). We show that every PI-ring which has no nilpotent ideals[2] is isomorphic to a subring of a complete direct sum[3] of central simple algebras whose order over their centre is bounded. As a consequence we prove that these rings are subrings of matrix rings over commutative rings. This implies an extension of a result of [2] concerning the minimal identity of a simple algebra. We prove that for a PI-ring which has no nonzero nilpotent ideals, the standard identity $S_d(x) = 0$, where d is an even integer, is the unique (up to a numerical factor) minimal identity which is linear in each of its indeterminates. The term *standard identity* was ascribed in [2] to the polynomial identity:

$$S_d(x) = S_d(x_1, \cdots, x_d) = \sum_{(i)} \pm x_{i_1} \cdots x_{i_d} = 0$$

where the sum ranges over all permutations (i) of d letters, and the sign is positive for even permutations and negative for odd permutations.

Notations. A polynomial identity of minimum degree satisfied by a PI-ring R will be called a *minimal identity* of R. We shall refer to a polynomial identity which is linear and homogeneous in each of its indeterminates as a *linear identity*. We shall use the following three types of semi-simplicity: a ring R is said to be

(a) J-semi-simple, if R is semi-simple in the sense of Jacobson [3], that is, if the quasi-regular radical of R is zero.

(b) K-semi-simple, if R does not contain any nonzero nil ideals.

(c) A-semi-simple, if R has no nonzero nilpotent ideals.

2. **The ring** $R[x]$. We denote by $R[x]$ the ring of all polynomials in the commutative indeterminate x over R. In this section we deal with properties of $R[x]$ induced by R.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Ideals will always mean two-sided ideal.

[3] For definition of (complete) direct sums and of subdirect sums see, for example, [1, p. 121].

LEMMA 1. *Let P be a nonzero ideal in $R[x]$ and let $p(x)=a_0+\cdots+a_nx^n$ $(a_n\neq 0)$ be a polynomial of minimum degree in P. Then if $b\in R$ such that $a_n^\mu b=0$ for some integer μ, then $a_n^{\mu-1}p(x)b=0$.*

Indeed, the coefficient of x^n in $a_n^{\mu-1}p(x)b\in P$ is $a_n^\mu b=0$, that is, this polynomial is of lower degree than that of $p(x)$. Hence the minimality of the degree of $p(x)$ implies that $a_n^{\mu-1}p(x)b=0$.

COROLLARY. *If $r(x)\in R[x]$ such that $a_n^\mu r(x)=0$ for some integer μ, then $a_n^\lambda p(x)r(x)=0$ for every integer $\lambda\geqq\mu-1$.*

This follows immediately by the preceding lemma, since each of the coefficients of $r(x)$ satisfies the condition of that lemma.

We prove now the following fundamental lemma:

LEMMA 2.[4] *If R is a* K*-semi-simple ring, then $R[x]$ is* J*-semi-simple.*

PROOF. Assume that $R[x]$ is not J-semi-simple. Denote by J_x the nonzero Jacobson's radical of $R[x]$. It is readily verified that the totality of the coefficients of the highest power of the polynomials of J_x of degree n—where n is the minimal degree of the nonzero polynomials of J_x—constitute a nonzero ideal in R. The lemma will be proved if it is shown that this ideal is a nil ideal, that is, that if $p(x)=a_0+a_1x+\cdots+a_nx^n$ is a nonzero polynomial of minimum degree in J_x, then $a_n^\mu=0$ for some integer μ.

To this end we consider the polynomial $p(x)xa_n$ (which belongs to J_x, since $p(x)\in J_x$ and $xa_n\in R[x]$) and its quasi-inverse $q(x)$. By Lemma 1 of [3] and Theorem 2 of [3] it follows that

$$p(x)xa_n+q(x)+p(x)xa_nq(x)=0, \tag{1}$$

$$p(x)xa_n+q(x)+q(x)p(x)xa_n=0. \tag{2}$$

By (1) we obtain that $q(x)=xt(x)$,[5] $t(x)\in R[x]$. Put $s(x)=p(x)a_n$. Then (1) implies that $xs(x)+xt(x)+x^2s(x)t(x)=0$. Hence,[6]

$$s(x)+t(x)+xs(x)t(x)=0. \tag{3}$$

Similarly, we obtain from (2) that

$$s(x)+t(x)+xt(x)s(x)=0. \tag{4}$$

Suppose $a_n^\mu t(x)\neq 0$ for every integer μ. Let ν be the minimal degree of the polynomials $a_n^\mu t(x)$. Write

[4] If R is commutative, this lemma is a consequence of [7, Corollary 8.1].

[5] If R does not possess a unit and $x\notin R[x]$, we adopt the notation $xt(x)$ (similarly $t(x)x$) for the polynomial $xb_0+\cdots+x^{n+1}b_n$, where $t(x)=b_0+\cdots+x^nb_n$.

[6] Since $xm(x)=0$ if and only if $m(x)=0$.

$$t(x) = t_1(x) + x^{\nu+1}t_2(x), \tag{5}$$

where $t_1(x) = b_0 + b_1x + \cdots + b_\nu x^\nu$. The minimality of ν implies that

$$a_n^\mu b_\nu \neq 0 \qquad \text{for every integer } \mu, \tag{6}$$

and

$$a_n^\mu t_2(x) = 0 \qquad \text{for every } \mu \text{ greater than some integer } \pi. \tag{7}$$

The polynomial $s(x) = p(x)a_n$ is of minimum degree in J_x, and its highest coefficient is a_n^2. Hence, since $a_n^{2\mu}t_2(x) = 0$ (for $\mu \geqq \pi$), it follows by the corollary of Lemma 1 that

$$a_n^\mu s(x)t_2(x) = 0 \qquad \text{for every } \mu \geqq 2\pi. \tag{8}$$

Substituting (5) into (3) and multiplying this equation on the left by a_n^λ, where $\lambda = 2\pi$, we obtain, by (7) and (8),

$$a_n^\lambda s(x) + a_n^\lambda t_1(x) + xa_n^\lambda s(x)t_1(x) = 0.$$

The degree of both $a_n^\lambda s(x)$ and $a_n^\lambda t_1(x)$ is less than $n+\nu+1$, and the coefficient of $x^{n+\nu+1}$ of $xa_n^\lambda s(x)t_1(x)$ is $a_n^{\lambda+2}b_\nu$. Hence $a_n^{\lambda+2}b_\nu = 0$. But this contradicts (6); hence our assumption that $a_n^\mu t(x) \neq 0$, for every integer μ, is false. Thus $a_n^\lambda t(x) = 0$ for some integer λ. Now multiplication of (4) on the left by a_n^λ yields $a_n^\lambda s(x) = 0$; hence $a_n^{\lambda+2} = 0$, q.e.d.

3. A-semi-simple PI-rings.

LEMMA 3. *If R is a* PI-*ring, then $R[x]$ is also a* PI-*ring, and the totalities of the linear-identities of R and $R[x]$, respectively, coincide.*

The first part of the lemma follows from the fact that R satisfies a linear identity (Lemma 2 of [4]), and this identity is evidently satisfied by $R[x]$. If we assume that the operators of R, which are the coefficients of the identities of R, were extended to operate on $R[x]$ by defining $\alpha(\sum a_\nu x^\nu) = \sum(\alpha a_\nu)x^\nu$, the rest of the lemma is readily verified.

The following lemma follows immediately:

LEMMA 4. *A necessary and sufficient condition that a subdirect sum of a set of PI-rings $\{Q_\alpha\}$ satisfies an identity $F(x_1, \cdots, x_m) = 0$ is that each of the rings Q_α satisfies the identity $F = 0$.*

We recall that a PI-ring R is said to be of degree d [5] if d is the minimal degree of the polynomial identities satisfied by R.

REMARK. It has been shown in [2] that a central simple algebra A of order n^2 over its centre is a PI-ring of degree $2n$, and the minimal linear-identity of A is the standard identity $S_{2n}(x)=0$, uniquely determined up to a numerical factor. Evidently, A satisfies also the identities $S_n(x)=0$ for every $m \geqq 2n$.[7]

We prove now:

THEOREM 1. *If R is a* J*-semi-simple PI-ring of degree d, then*

(1) $d=2m$.

(2) *The ring R is a subdirect sum of a set of central simple algebras $\{A_\alpha\}$ such that m^2 is the upper bound of the orders of these algebras over their centres.*

(3) *The standard identity $S_d(x)=0$ is the unique* (*up to a numerical factor*) *minimal linear-identity of R.*

PROOF. Since R is J-semi-simple, R is a subdirect sum of primitive rings $\{A_\alpha\}$ (Theorem 28 of [3]), Lemma 4 implies that each A_α is a PI-ring of degree not greater than d. Hence, by Theorem 1 of [4] and by consequence 2 of [5] it follows that each A_α is a central simple algebra of order not greater than $[d/2]^2$. Let m^2 be the upper bound of the orders of the algebras A_α; then $m \leqq [d/2]$. By the preceding remark it follows that each A_α satisfies the identity $S_{2m}(x)=0$. Thus, Lemma 4 implies that this identity is satisfied, as well, by their subdirect sum R; hence, $d \leqq 2m$. On the other hand, $2m \leqq 2[d/2] \leqq d$. Hence $m=[d/2]$ and $d=2m$. This completes the proof of the first two parts of the theorem. Since the upper bound m^2 is achieved by some A_β, and the minimal identities of R, whose degree is $2m$, are also identities of this algebra, the proof of the third part of our theorem follows immediately by the preceding remark, that is, by Theorem 7 of [2].

We turn now to the main theorem of this paper:

THEOREM 2. *Let R be an* A*-semi-simple PI-ring of degree d, then*

(1) $d=2m$.

(2) *The ring R is a subring of a complete direct sum of central simple algebras $\{A_\alpha\}$ such that m^2 is the upper bound of the orders of these algebras over their centres.*

(3) *The identity $S_d(x)=0$ is the unique* (*up to a numerical factor*) *minimal linear-identity of R.*

PROOF. Since R is a PI-ring which is A-semi-simple, the corollary of Theorem 4 of [5] implies that R is also K-semi-simple; hence by Lemma 2 it follows that $R[x]$ is J-semi-simple.

In the light of Lemma 3, the application of the preceding theorem to the ring $R[x]$ yields the first and the third parts of the theorem.

[7] Compare with Remark 6 of [2].

The rest of the theorem follows now immediately from the preceding theorem since R is a subring of $R[x]$ which is, by Lemma 3, a PI-ring of degree d.

Let $R[x]$ be a subdirect sum of the central simple algebras $\{A_\alpha\}$. By Lemma 4 it follows that the set of the identities satisfied by every A coincides with the set of the identities of the complete direct sum $\sum A_\alpha$ as well as with the totality of the identities of $R[x]$. Hence we obtain, by Lemma 3, the following corollary.

COROLLARY 1. *The set of the linear identities of the* PI-*ring R is the same as the set of the linear identities of the complete direct sum* $\sum A_\alpha$.

Let $\{A_\alpha\}$ be a set of central simple algebras of orders not greater than m^2. Then each of these algebras satisfies the identity $S_{2m}(x)=0$. Lemma 3 implies, therefore, that the complete direct sum $\sum A_\alpha$ satisfies the same identity $S_{2m}(x)=0$. A combination of this fact and the preceding theorem yields:

COROLLARY 2. *A necessary and sufficient condition for an* A-*semi-simple ring to satisfy a polynomial identity is that it be isomorphic to a subring of a complete direct sum of central simple algebras of bounded order.*

Another immediate consequence of the preceding theorem is:

COROLLARY 3. *Every PI-ring of odd degree contains nonzero nilpotent ideals.*

Consider the ring R and the central simple algebras A_α of Theorem 2. Let F_α be a splitting field of the algebra A_α. Then A_α is isomorphic with a subring of the total matrix algebra $F_{\alpha m}$ of order m^2 over F_α. The complete direct sum $\sum F_{\alpha m}$ of the matrix algebra $\{F_{\alpha m}\}$ contains, therefore, a subring isomorphic with the complete direct sum $\sum A_\alpha$. Thus it follows by Theorem 2 that R is isomorphic with a subring of $\sum F_{\alpha m}$. It is readily verified that $\sum F_{\alpha m}$ is isomorphic with the total matrix ring F_m of order m^2 over the complete direct sum $F=\sum F_\alpha$ of the fields $\{F_\alpha\}$. Since F is a direct sum of fields, F is a commutative A-semi-simple ring. Hence, we obtain:

THEOREM 3.[8] *If R is a PI-ring of degree d without nilpotent ideals, then $d=2m$ and R is isomorphic with a subring of a total matrix ring of order m^2 over a commutative ring which does not contain nilpotent ideals.*

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[8] This result has been pointed out to me by the referee.

Let R be a subring of a total matrix ring of order m^2 over a commutative ring. By the proof of [2, Theorem 1] it follows that R is a PI-ring which satisfies the identity $S_{2m}(x)=0$. Hence, a combination of this fact and the preceding theorem yields:

COROLLARY. *An* A-*semi-simple ring R is a PI-ring if and only if R is isomorphic with a subring of a total matrix ring over a commutative ring.*

4. **Identities for PI-rings.** Denote by $N=N(R)$ the radical of the PI-ring R, that is, the join of all nilpotent ideals of R.

In this section we apply the preceding results to obtain identities satisfied by the quotient ring $R/N(R)$.[9]

Let R be a PI-ring of degree d, and let $U(R)$ denote the lower radical of R. Since $R/U(R)$ is an A-semi-simple PI-ring, it follows by Theorem 2, that:

THEOREM 4. *If R is a PI-ring of degree d, and $U(R)$ is the lower radical of R, then $R/U(R)$ satisfies the identity $S_{2m}(x)=0$, where $2m\leqq d$.*

THEOREM 5. *Let R be a PI-ring of degree d such that its radical $N(R)$ is a nilpotent ideal of index not greater than ρ, then S satisfies the identity*

$$\prod_{i=1}^{\rho} S(x_{i1}, \cdots x_{id}) = 0. \tag{9}$$

PROOF. The condition of the theorem implies that $U(R)=N(R)$. Hence, by the preceding theorem, $R/N(R)$ satisfies each of the identities $S(x_{i1}, \cdots, x_{id})=0$. Since $N(R)^\rho=0$, it is readily seen that R satisfies the identity (9).

By Theorem 2 of [6] it follows that the radical of the quotient ring $R/N(R)$, where R is a PI-ring of degree d, is a nilpotent ideal of index not greater than $[d/2]$. Hence we have the following corollary.

COROLLARY. *If R is a PI-ring of degree d, then $R/N(R)$ satisfies the identity $\prod_{i=1}^{[d/2]} S(x_{i1}, \cdots, x_{id})=0$.*

In a process similar to that of the Laplace expansion of determinants one can readily prove that

$$S_n(x_1, \cdots, x_n) = \sum \pm S_k(x_{i_1}, \cdots, x_{i_k}) S_{n-k}(x_{i_{k+1}}, \cdots, x_{i_n})$$

where the sum ranges over all $C_{n,k}$ different selections of k letters $i_1, \cdots, i_k$ out of n letters, and where $i_{k+1}, \cdots, i_n$ denotes the complement of the set $i_1, \cdots, i_k$. This readily implies that the standard

[9] Compare with Theorem 9 and its remark of [2].

identity $S_{pq}(x)=0$ can be expressed as a sum of a set of q products of standard identities each of which is of degree p. Hence by the preceding corollary it follows that:

THEOREM 6. *If R is a PI-ring of degree d, then $R/N(R)$ satisfies the standard identity $S_p(x)=0$, where $p=d[d/2]$.*

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