NEIGHBORLY FUNCTIONS

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1. **Introduction.** In the following ρ metrizes S and ρ' metrizes S'. We agree[1] that a function f is *neighborly at the point* x if and only if for each $\epsilon > 0$ there exists[2] an open sphere α of S such that $\rho(x, y) + \rho'(f(x), f(y)) \leqq \epsilon$ whenever $y \in \alpha$. We also agree that a function is *neighborly* if and only if it is neighborly at each point of S. Obviously every continuous function is neighborly.

It is well known that if g is a function on S to S' and if f is such a sequence of continuous functions that $\lim_{n\to\infty} \rho'(f_n(x), g(x)) = 0$ for each x in S, then the points of discontinuity of g form a set of first ρ category. It is the principal purpose of the present note to show that this same conclusion can be drawn when the approximating functions are merely restricted to being neighborly.

2. THEOREM. *If g is a function on S to S' and f is such a sequence of neighborly functions that* $\lim_{n\to\infty} \rho'(f_n(x), g(x)) = 0$ *for each x in S, then the points of discontinuity of g form a set of first ρ category.*

PROOF. Let $w(x) = \lim \sup_{y\to x} \rho'(g(x), g(y))$ for x in S. Since the set of points of discontinuity of g is the set where $w(x) > 0$, the desired conclusion is a consequence of the following statement.

STATEMENT. If n is a non-negative integer and $0 < \epsilon < \infty$ and $A = SEx\ (w(x) \geqq \epsilon$ and $\rho'(f_m(x), g(x)) \leqq \epsilon/16$ for each integer $m \geqq n)$, then A is ρ nondense.

PROOF. Suppose

$$(1) \qquad A \text{ is dense in some open sphere } \alpha.$$

Let $x_1 \in A\alpha$ and use the neighborliness of f_n to secure such an open sphere α_1 that $\alpha_1 \subset \alpha$ and

$$(2) \qquad \rho'(f_n(x_1), f_n(z)) \leqq \epsilon/16 \qquad \text{whenever } z \in \alpha_1.$$

Let $x \in \alpha_1$ and choose such an integer m that $m \geqq n$ and

$$(3) \qquad \rho'(f_m(x), g(x)) \leqq \epsilon/16.$$

Now use the neighborliness of f_m to secure such an open sphere α_2 that $\alpha_2 \subset \alpha_1$ and

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[2] It should be noted that we do *not* require that $x \in \alpha$.

$$\rho'(f_m(x), f_m(z)) \leqq \epsilon/16 \qquad \text{whenever } z \in \alpha_2. \tag{4}$$

Let $x_2 \in A\alpha_2$. From (3), (4), (2) and the fact that $x_2 \in A$ and $x_1 \in A$, it follows that

$$\begin{aligned} \rho'(g(x), g(x_1)) &\leqq \rho'(g(x), f_m(x)) + \rho'(f_m(x), f_m(x_2)) \\ &\quad + \rho'(f_m(x_2), g(x_2)) + \rho'(g(x_2), f_n(x_2)) \\ &\quad + \rho'(f_n(x_2), f_n(x_1)) + \rho'(f_n(x_1), g(x_1)) \\ &\leqq \epsilon/16 + \epsilon/16 + \epsilon/16 + \epsilon/16 + \epsilon/16 + \epsilon/16 \\ &= 3\epsilon/8. \end{aligned}$$

Thus $\rho'(g(x), g(x_1)) \leqq 3\epsilon/8$ whenever $x \in \alpha_1$. Accordingly $\rho'(g(x), g(y)) \leqq 3\epsilon/4$ whenever $x \in \alpha_1$ and $y \in \alpha_1$. Thus $w(x) \leqq 3\epsilon/4$ whenever $x \in \alpha_1$, $A\alpha_1$ is empty and, in contradiction to (1), A is not dense in α. Therefore A is ρ nondense.

3. Remark. In order to appraise the generality of Theorem 2 we agree that f is ***neighborly' at the point*** x if and only if for each $\epsilon > 0$ there exists a sphere α of S such that $\rho(x, y) + \rho'(f(y), f(z)) \leqq \epsilon$ whenever $y \in \alpha$ and $z \in \alpha$. We further agree that f is ***neighborly'*** if and only if it is neighborly' at each point of S. Now it is clear that any neighborly function is neighborly'. Moreover the set of points of discontinuity of any neighborly' function is of the first ρ category. However, it is easily seen that the statement resulting from Theorem 2 by replacing "neighborly" by "neighborly'" is not a theorem.

4. Remark. The function f on the real numbers for which

$$f(x) = \begin{cases} \sin 1/x & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

is neighborly. Also, if for each real x, f is continuous from the right or from the left at x, then f is a neighborly function.

It is easy to see that a function on the reals may be neighborly at a point but discontinuous everywhere. Furthermore it is possible to find a neighborly function which is in Baire's second class and yet discontinuous almost everywhere. It is also possible to find a neighborly function which is nonmeasurable relative to every measurable set of positive measure. Such a function is of course discontinuous almost everywhere.

The limit of a sequence of neighborly functions is not necessarily neighborly, but a uniform limit of neighborly functions is neighborly.

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