REMARKS ON "SOME PROBLEMS IN CONFORMAL MAPPING"

JAMES A. JENKINS

1. The present note contains several remarks on an earlier paper by the author [2].[1] In Chapter IV, §4, which deals with the question of when we can have equality of modules for a triply-connected domain and a proper subdomain, the last sentence was added in proof. This accounts for the apparent disparity between it and the preceding one. In order to justify this statement we observe first that equality of the modules for triply-connected domains implies equality of the modules for the hexagons into which the domains are divided by their lines of symmetry. Thus by Theorem 5 and the remarks preceding Theorem 2a in [2], the value of $a_1:a_2:a_3$ in question must give rise to a degenerate case for each domain. The canonical domains will be of type 2, 4, or 5, the cases rising being those indicated on p. 344 in [2]. For the triply-connected domains there is a symmetric mapping of the one into the other characterized by the type of the canonical domains in question. We observe that for the functions ζ defined by equations (1), (2), (3) on pp. 339–340 in [2] if we set $(d\zeta/dz)^2 = Q(z)$ and choose the constant C appropriately, then $Q(z)dz^2$ is a positive quadratic differential of the triply-connected domain. The zeros of this quadratic differential lie at z^* and its symmetric point in cases (1), (2), and at z^*, z^{**} in case (3).

If we count boundary zeros with half their multiplicity, the total multiplicity of the zeros is 2 in each case (as also would follow from the general theory). Indeed these give all the positive quadratic differentials of the triply-connected domain (see [3]). In the canonical domain of type 2 the point $\overline{A}$ in Fig. 3 corresponds to such a zero. In the canonical domain of type 4 the points corresponding to zeros are $\overline{A}$ and A_5 (for the case drawn) and in the canonical domain of type 5 they are A_5 and A_6 (for the case drawn). The cases where equality of modules may occur are those illustrated in Fig. 5. They correspond to the following situations in order: (i) The subdomain is obtained from the original domain by producing slits out from the zeros of $Q(z)dz^2$ into the domain along the curves on which $Q(z)dz^2 > 0$ (corresponding to the vertical segment through $\overline{A}$ in the domain in Fig. 3), these slits are symmetric with respect to the line of symmetry; (ii) the subdomain is obtained from the original domain by

Received by the editors April 25, 1951.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

producing a slit out from a boundary zero on the line of symmetry to the other zero (which is interior and must lie on the line of symmetry, $Q(z)dz^2$ being positive on this slit) and then having the slit fork out symmetrically along the other two curves on which $Q(z)dz^2 > 0$ passing through the latter zero, in this case the quadratic differential induced on the subdomain corresponds to a canonical domain of type 2; (iii) the subdomain is obtained from the original domain by producing a slit out from a boundary zero on the line of symmetry but not reaching out beyond the other zero (as it did above); (iv) the subdomain is obtained from the original domain by producing two slits out from boundary zeros on the line of symmetry (necessarily on the same connected piece of the line of symmetry in the domain). It is clear that any domain has all these types of subdomains giving equality of the modules for particular values of $a_1 : a_2 : a_3$. The statement made in [2] was that in the third and fourth cases the subdomain could not be mapped into the original domain in any other way subject to the given topological restrictions. This is not so in the first two cases but we are able to characterize all possible such mappings. Let us consider in each situation the metric $|Q(z)|^{1/2}|dz|$ where $Q(z)dz^2$ is the associated quadratic differential in the original domain. We shall call it for short the Q-metric and it corresponds to the Euclidean metric in the canonical domains. Now let us consider the first case above. We suppose we are in the particular situation illustrated in Fig. 5, that is, the zeros of $Q(z)dz^2$ are on $\mathfrak{K}_2$ and the subdomain is obtained by producing two symmetric slits out from them into D along the curves on which $Q(z)dz^2 > 0$. Then, for the subdomain, $\mathfrak{K}_1'$, $\mathfrak{K}_3'$ coincide with $\mathfrak{K}_1$, $\mathfrak{K}_3$. We may think of the domain D as represented by the canonical domain together with its reflection in its base with points on A_5A_6, A_3A_4 being identified respectively with their reflected images. To the vertical segment joining $\overline{A}$ with its reflected image corresponds in D a curve L joining the zeros of the quadratic differential and on which $Q(z)dz^2 > 0$. To the right of it the vertical segments give curves of class C_1 of precise length $2a_1$. To the left of it the vertical segments give curves of class C_3 of precise length $2a_3$. These are the only curves of their respective classes which have precisely this minimal length and we shall call them, for short, trajectories. We note that L divides D into two doubly-connected domains, say Δ_1 with $\mathfrak{K}_1$ on its boundary, Δ_3 with $\mathfrak{K}_3$ on its boundary, and that these curves are also trajectories for the module problems in these domains.

Now suppose that we have a different admissible mapping $\phi(z)$ of D' into D. The first possibility is that each trajectory goes into a curve having the same length, that is, $2a_1$ for curves in class C_1, $2a_3$ for curves in class C_3. By the remark above the image curve must again be a trajectory. In particular $\mathfrak{K}_1'$ must go into $\mathfrak{K}_1$, $\mathfrak{K}_3'$ must go into $\mathfrak{K}_3$ (otherwise the subdomain would have a strictly smaller module for all $a_1:a_2:a_3$). Consider a trajectory γ_1 in Δ_1. Since the conformal map ϕ must preserve the module of the doubly-connected domain bounded by $\mathfrak{K}_1$ and γ_1, γ_1 must go into itself. Similarly each trajectory in Δ_3 must go into itself. Moreover the mapping of Δ_1 could only be a "conformal rotation," that is, correspond to a rotation of the circular ring conformally equivalent to Δ_1. In the canonical domain representation this corresponds to a vertical translation modulo $2a_1$. Similar remarks apply to Δ_3 and, since the mapping ϕ must be continuous along L, the corresponding mapping in the canonical domain would be a vertical translation with proper regard to the identifications. Conversely any translation up or down by an amount less than the length of $\overline{AA'}$ (see Fig. 5) does give an admissible conformal map ϕ of D' into D different from the identity. Its image is obtained by lengthening one of the slits produced into D and shortening the other by the same amount in the Q-metric. We shall now see that these are the only possible admissible conformal maps of D' into D.

Indeed, the alternative would be that some trajectory went into a curve of greater length in the Q-metric. Say a trajectory γ_1 in Δ_1 does so, going into a curve of length $2a_1+2d$, $d>0$. Consider at a point of γ_1 a little curve segment on which $Q(z)dz^2<0$ (that is, on an orthogonal trajectory of the trajectories). Then by continuity for an interval on this curve sufficiently small in the Q-metric, say of Q-length c, the trajectories meeting it will have images of Q-length at least $2a_1+d$. The other trajectories go into curves of length at least equal to their own. We now compute a lower bound of the area of the image of D' in the Q-metric by the standard method [**2**, pp. 328–329 or pp. 331–332]. Indeed this is just the area of D' in the metric $\rho^*(z)|dz| = |Q(z)|^{1/2}|\phi'(z)||dz|$. For this we have for all γ_1 in Δ_1, γ_3 in Δ_3

$$\int_{\gamma_\cdot} \rho^*(z)\,|\,dz\,| \geqq 2a_1, \qquad \int_{\gamma} \rho^*(z)\,|\,dz\,| \geqq 2a_3$$

while for the special γ_1 in Δ_1 meeting the above interval of Q-length c

$$\int_{\gamma_\cdot} \rho^*(z)\,|\,dz\,| \geqq 2a_1 + d.$$

Integrating these in the orthogonal direction in the canonical domain we get ($d\sigma$ element of Euclidean area in D)

$$\int_{D'} \rho^*(z) \, | \, Q(z) \, |^{1/2} d\sigma \geqq 2a_1\beta_1 + 2a_3\beta_3 + cd,$$

where β_1, β_3 are the lengths of A_3A_4, A_5A_6 as usual. Then by the standard argument

$$\int_{D'} (\rho^*(z))^2 d\sigma \geqq 2a_1\beta_1 + 2a_3\beta_3 + 2cd.$$

However the Q-area of the image of D' cannot exceed the Q-area of D and thus we have a contradiction.

The second case illustrated in Fig. 5 is treated in essentially the same manner. All other conformal maps into D correspond to vertical translations in the canonical domain and the images are obtained by lengthening one prong of the fork and shortening the other by the same amount in the Q-metric. That there are no other mappings is proved in the same way as above.

In the third and fourth cases the discussion proceeds in the same manner up to the point where the question of the "conformal rotation" of the doubly-connected domains arises. This is now impossible in view of the mapping ϕ being continuous along the segment of the line of symmetry corresponding to $A_5'A_6'$ (or at A_5' in the extreme case that this coincides with $\overline{A}$ in the third case). Hence there are now no admissible mappings ϕ other than the identity in which each trajectory goes into a trajectory of the same length. That there are no admissible mappings where this fails follows in the same way as above. This completes the discussion.

2. We observe further that for the conformal equivalence of hexagons (and thus of triply-connected domains) it is sufficient to know

$$M(1, 0, 0) = M'(1, 0, 0), \qquad M(0, 1, 0) = M'(0, 1, 0),$$
$$M(0, 0, 1) = M'(0, 0, 1).$$

Indeed if we map each hexagon on a half-plane the modules of the quadrangles with vertices 1, 2, 3, 4; 3, 4, 5, 6; 5, 6, 1, 2 are given in a unique monotone manner in terms of the cross ratios of their vertices [**1**, p. 343]. A similar remark is true for the vertices $1'$, $2'$, $3'$, $4'$, $5'$, $6'$ of the second hexagon. Thus we can make a linear transformation bringing $1'$, $2'$, $3'$, $4'$ into coincidence respectively with 1, 2, 3, 4. Further we may suppose that the half-plane is bounded by the real axis and that these points have abscissae ∞, 0, 1, x $(x>1)$. Then an elementary calculation shows that, as a consequence of equality of cross ratios, 5 and 5′, 6 and 6′ must coincide. Thus the hexagons are conformally equivalent. We point out that three is the number of conformal moduli of a hexagon or a triply-connected domain.

For a pentagon even more is true. There the conditions

$$M(1, 0) \geqq M'(1, 0), \qquad M(0, 1) \geqq M'(0, 1)$$

are sufficient that the second pentagon may be mapped into the first subject to the assigned boundary conditions. This can be derived from the considerations of [2, chap. II] or proved directly in the same manner as above (as was pointed out to me by A. Dvoretzky).

Bibliography

1. L. Ahlfors and A. Beurling, *Invariants conformes et problèmes extrémaux*, Dixième Congrès des Mathématiciens Scandinaves, 1946.

2. J. A. Jenkins, *Some problems in conformal mapping*, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 327–350.

3. ———, *Positive quadratic differentials in triply-connected domains*, Ann. of Math. vol. 53 (1951) pp. 1–3.

The Johns Hopkins University