COMPARISON OF TOPOLOGIES ON FUNCTION SPACES

JAMES R. JACKSON

1. **Introduction.** Let X be a topological space, Y be a metrizable space, and C be the collection of continuous functions on X into Y. We shall be interested in two topologies on C.

The compact-open or *k-topology*. Given compact subset K of X and open subset W of Y, denote by (K, W) the collection of functions $f \in C$ such that $f(K) \subset W$. The assembly of finite intersections of sets (K, W) forms a base for a topology on C, called the compact-open topology, or k-topology [1].[1] Here the restriction to metrizable spaces Y is unnecessary.

The topology of uniform convergence or d^*-topology. If Y is metrizable, let d be a bounded metric consistent with the topology of Y. For instance, if d' is an unbounded metric, let

$$d(y_1, y_2) = \min \{d'(y_1, y_2); 1\}, \qquad y_1, y_2 \in Y.$$

Then for any two elements f and g of C, define

$$d^*(f, g) = \sup d(f(x), g(x)) \qquad \text{over } X.$$

It is easily verified that d^* is a metric function, and hence determines a topology on C. This topology is called the topology of uniform convergence with respect to d, or d^*-topology [3].

Arens [1] and Fox [2] have shown that the k-topology has certain properties which particularly adapt it to the study of various topological problems, particularly those concerned with homotopy theory. But in case Y is metrizable, so that the d^*-topology can be defined, it is obviously easier to deal with than the k-topology. Hence it is of interest to inquire when the two topologies are equivalent.

It is easily shown that if Y is metrizable and X is compact, then the d^*-topology on C is equivalent to the k-topology (see, for instance, [3]). On the other hand, a theorem of Fox [2] implies that if X is a separable metric space which is not locally compact and Y is the real line, then the two topologies are inequivalent. Hu [3] gives an example due to Liang Ma wherein X is a countable discrete space, and the two topologies are inequivalent. We shall answer in a fairly general way the question of the relationship between these topologies.

Let Y be metrizable. We shall show in §2, without further restric-

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[1] Numbers in brackets refer to the references cited at the end of the paper.

tion on X or Y, that the d^*-topology is weaker than the k-topology (that is, every set open under the k-topology is also open under the d^*-topology). As remarked above, the compactness of X is a sufficient condition for the equivalence of the two topologies. We shall show in §3 that if X is completely regular and Y contains a nondegenerate arc, then the compactness of X is also a necessary condition for the equivalence of the d^*-topology and the k-topology.

2. THEOREM. *If Y is metrizable, then the d^*-topology is weaker than the k-topology.*

We must show that if U is open under the k-topology and $f \in U$, then we can find $\epsilon > 0$ such that if $d^*(f, g) < \epsilon$, then $g \in U$.

By hypothesis, there exist compact subsets of X, $K_1, \cdots, K_n$, and open subsets of Y, $W_1, \cdots, W_n$, such that

$$f \in (K_1, W_1) \cap \cdots \cap (K_n, W_n) \subset U.$$

Since each K_i is compact, so is each $f(K_i)$. Also, each $f(K_i)$ is disjoint from the corresponding closed set $Y - W_i$. But the distance between two disjoint subsets of metric space, one of which is compact and the other closed, is positive. Let ϵ be a positive number less than the smallest of the distances from the $f(K_i)$ to the corresponding $Y - W_i$.

Now suppose $d^*(f, g) < \epsilon$. If $x \in K_i$, then the distance from $g(x)$ to $f(K_i)$ is less than ϵ, so $g(x) \in W_i$. Thus

$$g \in (K_1, W_1) \cap \cdots \cap (K_n, W_n) \subset U,$$

so $g \in U$, as required.

3. THEOREM. *Let X be a completely regular space, and let Y be a metrizable space containing a nondegenerate arc. Then a necessary condition that the d^*-topology and the k-topology be equivalent is that X be compact.*

Let ϕ: $I \rightarrow Y$ define a nondegenerate arc, where I denotes the closed unit interval. Pick $t_1 \in I$ so $y_1 = \phi(t_1)$ is different from $y_0 = \phi(0)$. Choose $\epsilon > 0$ so $d(y_0, y_1) \geqq \epsilon$. Define f as the function of C which is constantly equal to y_0.

Suppose the two topologies are equivalent. Then there must be compact subsets of X, $K_1, \cdots, K_n$, and open subsets of Y, $W_1, \cdots, W_n$, such that

$$f \in (K_1, W_1) \cap \cdots \cap (K_n, W_n) \subset S,$$

where S is the set of functions $g \in C$ such that $d^*(f, g) < \epsilon$.

We complete the proof by contradiction. If $X - (K_1 \cup \cdots \cup K_n)$

is empty, then X is the union of a finite number of compact sets, and hence must be compact.

Otherwise, choose $x_0 \in X-(K_1 \cup \cdots \cup K_n)$. Since x_0 and $(K_1 \cup \cdots \cup K_n)$ are disjoint closed subsets of the completely regular space X, there exists a continuous function θ: $X \rightarrow I$ with $\theta(x)=0$ on $(K_1 \cup \cdots \cup K_n)$ and $\theta(x_0)=t_1$. Define g: $X \rightarrow Y$ as the product $\phi\theta$; $g(x)=\phi(\theta(x))$. It is obvious that $g \in C$. Moreover, for $x \in K_i$, we have that $g(x)=\phi(\theta(x))=\phi(0)=y_0 \in W_i$; so $g(K_i) \subset W_i$, and $g \in (K_1, W_1) \cap \cdots \cap (K_n, W_n)$.

On the other hand,

$$\begin{aligned} d^*(f, g) \geqq d(f(x_0), g(x_0)) &= d(y_0, \phi(\theta(x_0))) \\ &= d(y_0, \phi(t_1)) \\ &= d(y_0, y_1) \geqq \epsilon. \end{aligned}$$

Hence $g \notin S$. This contradiction completes the proof.

4. **A generalization.** If instead of requiring Y to be metrizable, we insist only that it be a uniform space, a topology of uniform convergence can be defined on C. In this case the compactness of X is a sufficient condition for the equivalence of the topology of uniform convergence and the k-topology, the obvious analogue of Theorem 2 holds without further restriction, and the analogue of Theorem 3 holds if we require that Y have a separated uniform structure. The proofs are not essentially different from those above, but are somewhat longer.

References

1. R. Arens, *A topology for spaces of transformations*, Ann. of Math. vol. 47 (1946) pp. 480–495.

2. R. Fox, *On topologies for function spaces*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 429–432.

3. S. T. Hu, *Homotopy theory*, vol. 1 (Dittoed typescript prepared at Tulane University under the auspices of the Office of Naval Research), 1950.

University of California, Los Angeles