COMMUTATOR CALCULUS AND LINK INVARIANTS

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1. Introduction. Let G be a finitely presented group such that the abelianized group $G/[G, G]$ has a basis of n elements, which may possibly include elements of finite order. Let G have a presentation of $n+k$ generators and $k+q$ relations. (Necessarily, $q \geqq 0$.) Then, for each integer $d>0$, a group $\mathfrak{G}$ presented by n generators and q relations may be constructed such that $\mathfrak{G}/\mathfrak{G}_d$ is isomorphic with G/G_d, where $\mathfrak{G}_d$ and G_d are the dth lower central commutator subgroups of $\mathfrak{G}$ and G respectively. In the case that G is the group of a link L consisting of n components, $\mathfrak{G}$ is a group presented by n generators and n relations. This is quite a helpful reduction in the number of generators and relations in the presentation of G/G_d, which determines the finitely generated abelian factor groups G_i/G_{i+1}, $i=1, 2, \cdots, d-1$, and thus yields numerical invariants. This result is applied in §4 to obtain a geometrical interpretation of the factor group G/G_3 of the group G of a link L; G/G_3 is completely determined by the number of components of L and the linking numbers of the different pairs of components. In §5 two examples are given. In one of them, it is shown that the torsion numbers of G_3/G_4 are sufficient to distinguish between a certain sequence of links, each of which has vanishing linking number for each pair of its three components. In the other example, it is shown that the torsion numbers of G_4/G_5 may be used to distinguish between another sequence of links of two components with vanishing linking number. However, for the group G of a knot, the factor group G/G_d is finite cyclic for every $d \geqq 2$. The author is obliged for the valuable suggestions of R. H. Fox and R. C. Lyndon.

2. Terminology and preliminary. For any group G, denote by $[a, b]$ the commutator $aba^{-1}b^{-1}$, $a, b \in G$, and by $[A, B]$ the subgroup generated by all $[a, b]$, $a \in A$, $b \in B$. Define inductively $[a_1]=a_1$, and for $d \geqq 1$, $[a_1, a_2, \cdots, a_{d+1}]=[[a_1, a_2, \cdots, a_d], a_{d+1}]$, $a_i \in G$. Furthermore, set $G_1=G$ and, for $d \geqq 1$, $G_{d+1}=[G_d, G]$. The group G_d is called the dth lower central commutator subgroup of G and is the normal subgroup generated by all $[a_1, a_2, \cdots, a_d]$, $a_i \in G$.

Let M be a normal subgroup of G. Then we write $a \equiv b \bmod M$ if and only if $ab^{-1} \in M$. Without ambiguity, we shall denote the cosets $a_1M, a_2M, \cdots$ by $a_1, a_2, \cdots \bmod M$. For any homomorphism

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ϕ: $G \to G'$, $\phi(G_d) \subset G'_d$, and, consequently, $a \equiv b \bmod G_d$ implies $\phi(a) \equiv \phi(b) \bmod G'_d$.

It is known [5][1] that $u \in G_s$, $v \in G_t$ implies $[u, v] \in G_{s+t}$. By repeated use of this fact together with the identities

$$(1) \qquad \begin{aligned} [ab, c] &= a[b, c]a^{-1}[a, c], \\ [c, ab] &= [c, a]a[c, b]a^{-1}, \end{aligned}$$

we obtain the following lemma.

LEMMA 1. *If $a, a' \in G_s$, $b, b' \in G_t$, $a \equiv a' \bmod G_{s+1}$, and $b \equiv b' \bmod G_{t+1}$, then*

$$[a, b] \equiv [a', b'] \bmod G_{s+t+1}.$$

3. The main theorem.

MAIN THEOREM. *Let G be a finitely presented group such that the abelianized group G/G_2 has a basis of n elements. Suppose that G is presented by $n+k$ generators and $k+q$ relations. (It is implied that $k \geqq 0$ and $q \geqq 0$.) Then, for each $d \geqq 0$, there is a group $\mathfrak{G}$ presented by n generators and q relations such that*

$$G/G_d \cong \mathfrak{G}/\mathfrak{G}_d.$$

The case $q=0$ of this theorem has been proved by W. Magnus [6]. This section is devoted to a complete proof of this theorem, which will be restated later in a more constructive form (Theorem 1).

Let G have a presentation

$$G \cong \{\bar{a}_i,\ i = 1, 2, \cdots, n + k \,/\, \bar{r}_i,\ i = 1, 2, \cdots, k + q\},$$

that is, G is obtained from the free group F generated by $\bar{a}_i$, $i=1, 2, \cdots, n+k$, by introducing relations $\bar{r}_i(\bar{a}_1, \cdots, \bar{a}_{n+k}) \equiv e$, $i=1, 2, \cdots, k+q$. The elements $\bar{a}_i$ also represent a set of generators of the abelianized group G/G_2, which has a basis of n elements. We apply to the array $\{\bar{a}_1, \cdots, \bar{a}_{n+k}\}$ the following operation:

(A) $\bar{a}_i$ and $\bar{a}_j$ are interchanged,

(B) $\bar{a}_i$ is replaced by $\bar{a}_i^{\epsilon}\bar{a}_j^{m}$, $i \neq j$, $\epsilon = \pm 1$;

and to the array $\{\bar{r}_1, \cdots, \bar{r}_{k+q}\}$ the following operations:

(C) $\bar{r}_i$ and $\bar{r}_j$ are interchanged,

(D) $\bar{r}_i$ is replaced by $\bar{r}_i^{\epsilon}\bar{r}_j^{m}$, $i \neq j$, $\epsilon = \pm 1$.

Under these operations $\bar{a}_1, \cdots, \bar{a}_{n+k}$ and $\bar{r}_1, \cdots, \bar{r}_{k+q}$ will continue to be generators and relations for G as well as for the abelian-

[1] Numbers in brackets refer to the bibliography at the end of the paper.

ized group G/G_2. Let $r_i \equiv \prod_{j=1}^{n+k} a_j^{m_{ij}}$ mod F_2, $i=1, 2, \cdots, k+q$. To the operations (A), (B), (C), (D) there correspond elementary transformations on the matrix $\|m_{ij}\|$. From the classical theory of matrices, it follows that, after a finite number of operations (A), (B), (C), (D), the arrays $\{\bar{a}_1, \cdots, \bar{a}_{n+k}\}$ and $\{\bar{r}_1, \cdots, \bar{r}_{k+q}\}$ will become such that

(i) $\bar{a}_1, \cdots, \bar{a}_n$ form a basis of G/G_2,

(ii) $\bar{r}_i \equiv x_i \bar{a}_{n+i}^{-1}$ mod F_2, $i=1, 2, \cdots, k$, x_i being a word in $\bar{a}_1, \cdots, \bar{a}_n$ alone,

(iii) $\bar{r}_{k+i} \equiv e$ mod F_2, $i=1, 2, \cdots, q$.

Thus we are led to the following lemma.

LEMMA 2. *Let G be defined as in the main theorem. Then G has a presentation* $G \cong \{a_1, \cdots, a_n; b_1, \cdots, b_k / h_1, \cdots, h_k; r_1, \cdots, r_q\}$ *such that the relations* $h_i \equiv e$ *and* $r_i \equiv e$ *are in the following forms*:

(i) $h_i = u_i x_i b_i^{-1}$, $u_i \in F_2$, $x_i \in \mathfrak{F}$,

(ii) $r_i \in F_2$,

where F denotes the free group generated by a_i, $i=1, 2, \cdots, n$, *and* b_i, $i=1, 2, \cdots, k$, *and* $\mathfrak{F}$ *denotes the free group generated by* a_i, $i=1, 2, \cdots, n$, *alone.*

We may assume hereafter in this section that G has its presentation as given in this lemma. Observe that $a_1, \cdots, a_n$ form a basis in the abelianized group G/G_2. For simplicity, write the presentation of G as $G \cong \{a, b / h, r\}$. Let H be the normal subgroup generated by h_i, $i=1, 2, \cdots, k$, and R the normal subgroup generated by r_i, $i=1, 2, \cdots, q$. Then $G \cong F/H \cdot R$.

Denote by $\psi(w)$ the word obtained from w by replacing each b_i by $u_i x_i$, and $\phi(w)$ the word obtained from w by replacing each b_i by x_i, $i=1, 2, \cdots, k$. Then the substitution ψ: $F \rightarrow F$ is an endomorphism of F, and the substitution ϕ: $F \rightarrow \mathfrak{F}$ is a homomorphism of F onto $\mathfrak{F}$. Both ψ and ϕ leave $\mathfrak{F}$ elementwise fixed, and $\phi^2 = \psi\phi = \phi$. Moreover $h_i = \psi(b_i) b_i^{-1}$, and therefore $\psi(w) \equiv w$ mod H, $w \in F$. We observe that, in the presentation $G \cong \{a, b / h, r\}$, to replace r_i by $\psi(r_i)$ is a Tietze operation [8], and thus, by repeated use of ψ,

$$G \cong \{a, b / h, \psi^{d-2}(r)\}, \qquad d \geqq 2. \tag{2}$$

The notation $\psi^{d-2}(r)$ stands for the array $\psi^{d-2}(r_1), \cdots, \psi^{d-2}(r_q)$, and obvious notations of abbreviation similar to this will be often used. It will be an important technique in this paper to use the substitution ψ as a Tietze operation on generators and relations of G.

LEMMA 3. *If* $w_t \in F_t$, *then*

$$\psi^d(w_t) \equiv \phi\psi^{d-1}(w_t) \bmod F_{d+t}, \qquad d \geqq 1.$$

PROOF. For the generators a_i and b_i of F, we have $\psi(a_i)=a_i=\phi(a_i)$ and $\psi(b_i)=u_ix_i\equiv\phi(b_i)$ mod F_2. It is thus true that, for $w\in F$, $\psi(w)\equiv\phi(w)$ mod F_2. We prove the lemma for $d=1$ by induction on t. Assuming that the lemma holds for $t-1$, $t\geqq 2$, it follows from Lemma 1 that $\psi([w_{t-1}, w])=[\psi(w_{t-1}), \psi(w)]\equiv[\phi(w_{t-1}), \phi(w)]$ mod F_{t+1} $\equiv\phi([w_{t-1}, w])$ mod F_{t+1}, $w\in F$, $w_{t-1}\in F_{t-1}$. Since each element of F_t is a product of commutators of the form $[w_{t-1}, w]$, we conclude that, for $w_t\in F_t$, $\psi(w_t)\equiv\phi(w_t)$ mod F_{t+1}. The lemma now holds for $d=1$. Proceeding by induction on d, we assume that it holds for $d-1$ and any $t\geqq 1$. Then $\psi^{d-1}(w_t)\equiv\phi\psi^{d-2}(w_t)$ mod F_{d+t-1}, that is, $\psi^{d-1}(w_t)[\phi\psi^{d-2}(w_t)]^{-1}\in F_{d+t-1}$. By the validity of the lemma for $d=1$, $\psi(\psi^{d-1}(w_t)[\phi\psi^{d-2}(w_t)]^{-1})\equiv\phi(\psi^{d-1}(w_t)[\phi\psi^{d-2}(w_t)]^{-1})$ mod F_{d+t}. Since $\psi\phi=\phi^2=\phi$, we have $\psi^d(w_t)[\phi\psi^{d-2}(w_t)]^{-1}\equiv\phi\psi^{d-1}(w_t)[\phi\psi^{d-2}(w_t)]^{-1}$ mod F_{d+t}. Hence

$$\psi^d(w_t) \equiv \phi\psi^{d-1}(w_t) \bmod F_{d+t}.$$

THEOREM 1. *Let a group G have a presentation as given in Lemma 2: $G\cong\{a, b \,/\, h, r\}$. Then the group $\mathfrak{G}\cong\{a \,/\, \phi\psi^{d-3}(r)\}$ has the property*

$$\mathfrak{G}/\mathfrak{G}_d \cong G/G_d, \qquad d \geqq 3.$$

REMARK. In the presentation of $\mathfrak{G}\cong\{a \,/\, \phi\psi^{d-3}(r)\}$, each r_i belongs to F_2, $i=1, 2, \cdots, q$. If, for some i, r_i belongs to F_t, $2\leqq t\leqq d-1$, then we may replace the corresponding $\phi\psi^{d-3}(r_i)$ by $\phi\psi^{d-1-t}(r_i)$ in the presentation of $\mathfrak{G}$.

PROOF OF THE THEOREM. It follows from Lemma 3 that $\psi^{d-2}(r_i)\equiv\phi\psi^{d-3}(r_i)$ mod F_d, and from (2) that

$$G/G_d \cong \{a, b \,/\, h, \psi^{d-2}(r), F_d\};$$

consequently,

$$G/G_d \cong \{a, b \,/\, h, \phi\psi^{d-3}(r), F_d\}.$$

Notice that each $\phi\psi^{d-3}(r_i)$ belongs to $\mathfrak{F}$. Let $h_i' =\phi\psi^{d-2}(b_i)b_i^{-1}$ and $h'=\{h_1', h_2', \cdots, h_k'\}$. Since $\psi(w)\equiv w$ mod H, $w\in F$, we have $\psi^{d-1}(b_i)\equiv b_i$ mod H and, using Lemma 3, $h_i' =\phi\psi^{d-2}(b_i)b_i^{-1}\equiv\psi^{d-1}(b_i)b_i^{-1}$ mod $F_d\equiv e$ mod $H\cdot F_d$. Therefore by Tietze operations we may introduce new relations $h_i' \equiv e$ into the presentation of G/G_d:

$$G/G_d \cong \{a, b \,/\, h, h', \phi\psi^{d-3}(r), F_d\}.$$

Now $\phi\psi^{d-2}$ is the substitution which replaces every b_i in a word by $\phi\psi^{d-2}(b_i)$, and, due to the definition of h', we may replace h by $\phi\psi^{d-2}(h)$ in this presentation of G/G_d:

$$G/G_d \cong \{a, b \,/\, \phi\psi^{d-2}(h),\ h',\ \phi\psi^{d-3}(r),\ F_d\}.$$

Observe that, due to Lemma 3, $\psi^{d-1}(b_i) \equiv \phi\psi^{d-2}(b_i) \bmod F_d$, and $\phi\psi^{d-2}(h_i) = \phi\psi^{d-2}(\psi(b_i)b_i^{-1}) = \phi[\psi^{d-1}(b_i)\phi\psi^{d-2}(b_i)^{-1}] \equiv e \bmod F_d$. Again, by Tietze operations,

$$G/G_d \cong \{a, b \,/\, h',\ \phi\psi^{d-3}(r),\ F_d\},$$

and, using $h_i' \equiv e$, that is, $b_i \equiv \phi\psi^{d-2}(b_i)$, as the defining relation of each b_i, we have

$$G/G_d \cong \{a \,/\, \phi\psi^{d-3}(r),\ \mathfrak{F}_d\}.$$

Let $\mathfrak{G} \cong \{a \,/\, \phi\psi^{d-3}(r)\}$. Then

$$\mathfrak{G}/\mathfrak{G}_d \cong \{a \,/\, \phi\psi^{d-3}(r),\ \mathfrak{F}_d\},$$

and hence the theorem is proved.

4. **Application to link groups.** A link is the union of n mutually disjoint, oriented, simple closed curves $L_1, \cdots, L_n$ in Euclidean 3-space E. L_i is called the ith component of L. If each L_i is a polygon, then L is said to be polygonal. The fundamental group G of the complement $E-L$ is called the group of the link L.

Through the well known Wirtinger method [7], we may read off a presentation of the group G of a polygonal link L through its regular projection. Let $G \cong \{a_{ij} \,/\, r_{ij}\}$ $(i=1, 2, \cdots, n;\ j=1, 2, \cdots, k_i)$ be such a presentation, where to each crossing point Q_{ij} of the projection corresponds a relation $r_{ij} \equiv e$, $r_{ij} = b_{ij}a_{ij}b_{ij}^{-1}a_{ij+1}^{-1} = [b_{ij}, a_{ij}]a_{ij}a_{ij+1}^{-1}$ with $b_{ij} = a_{\alpha(i,j)\beta(i,j)}^{\epsilon_{ij}}$. ($\alpha(i, j)$ and $\beta(i, j)$ are given by the segment of L which crosses over at Q_{ij}, and $\epsilon_{ij} = \pm 1$ is the signature of crossing.) $a_{i1}, a_{i2}, \cdots, a_{ik_i}$ are the Wirtinger generators corresponding to the segments (in their natural order) of the component L_i. The index j on a_{ij}, b_{ij}, $\cdots$, and so on, is to be taken modulo k_i.

Define $a_i = a_{i1}$, $v_{ij} = [b_{ij}, a_{ij}]$, $r_i = v_{ik_i}v_{ik_i-1} \cdots v_{i1}$, $u_{i1} = e$, $i=1, 2, \cdots, n$, $j=1, 2, \cdots, k_i$. Let $u_{ij} = v_{ij-1}v_{ij-2} \cdots v_{i1}$, $i=1, 2, \cdots, n$; $j=2, 3, \cdots, k_i$. Define $h_{ij} = u_{ij}a_ia_{ij}^{-1}$. It may be straightforwardly verified that

$$G \cong \{a_{ij} \,/\, h_{ij},\ r_i\}, \qquad i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, k_i.$$

Each r_i belongs to F_2. Define $\mathfrak{G} \cong \{a_i \,/\, \phi\psi^{d-3}(r_i)\}$, $i=1, 2, \cdots, n$, where $d \geqq 3$, $\psi(a_{ij}) = u_{ij}a_i$, and $\phi(a_{ij}) = a_i$. Then, due to Theorem 1, we have $G/G_d \cong \mathfrak{G}/\mathfrak{G}_d$.

In the case $d=3$, we have $\mathfrak{G} \cong \{a_i \,/\, \phi(r_i)\}$, $i=1, 2, \cdots, n$. Let $\mathfrak{F}$ be the free group generated by a_i, $i=1, 2, \cdots, n$; then $\mathfrak{G}/\mathfrak{G}_3 \cong \{a_i \,/\, \phi(r_i),\ \mathfrak{F}_3\}$. Define $r_i^* = \prod_{j=1, i\neq j}^{n} [a_j, a_i]^{\mu_{ij}}$, $i=1, 2, \cdots, n$, where μ_{ij} is the linking number of L_i and L_j. Then $\phi(r_i) = [\phi(b_{ik_i}), \phi(a_{ik_i})] \cdots [\phi(b_{i1}), \phi(a_{i1})] = [a^{\epsilon_{ik_i}}_{\alpha(i,k_i)}, a_i] \cdots [a^{\epsilon_{i1}}_{\alpha(i,1)}, a_i] \equiv r_i^* \bmod \mathfrak{F}_3$ and $\mathfrak{G}/\mathfrak{G}_3 \cong \{a_i / r_i^*, \mathfrak{F}_3\}$. Thus we have shown the following theorem.

THEOREM 2. *Let $L = L_1 \cup \cdots \cup L_n$ be a polygonal link, and G its group. Let μ_{ij} be the linking number of L_i and L_j, $i \neq j$. Define $\mathfrak{G}^* = \{a_i / r_i^*\}$, where $i = 1, 2, \cdots, n$, and $r_i^* = \prod_{j=1, j\neq i}^{n} [a_j, a_i]^{\mu_{ij}}$. Then $\mathfrak{G}^*/\mathfrak{G}_3^*$ is isomorphic with G/G_3.*

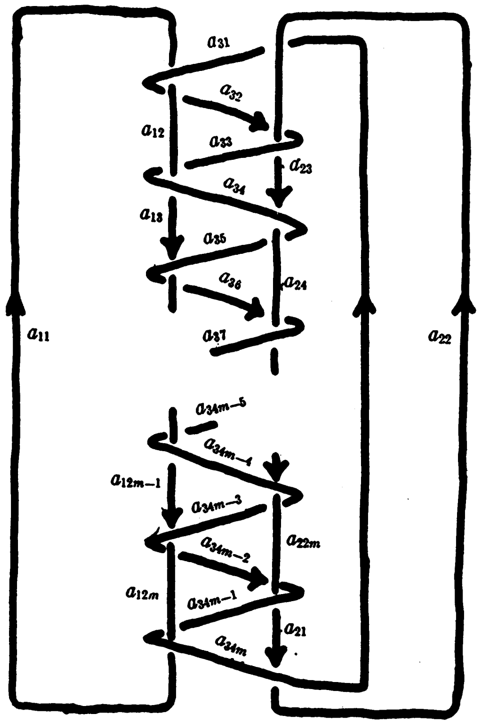

FIG. 1

COROLLARY 1. *$G/[G, G]$ is free abelian of rank n.*

COROLLARY 2. *$[G, G]/[[G, G], G] (= G_2/G_3)$ is isomorphic with an additive group generated by x_{ij}, $i \neq j$, $i, j = 1, 2, \cdots, n$, with relations*

(a) $$x_{ij} + x_{ji} = 0, \ i \neq j, \qquad i, j = 1, 2, \cdots, n,$$

(b) $$\sum_{j=1, i\neq j}^{n} \mu_{ij} x_{ij} = 0, \qquad i = 1, 2, \cdots, n.$$

PROOF. Let $\mathfrak{R}^*$ be the normal subgroup generated by all r_i^*,

$i=1, 2, \cdots, n$, in $\mathfrak{F}$. We have $\mathfrak{R}^* \subset \mathfrak{F}_2$. It is straightforward that

$$G_2/G_3 \cong \mathfrak{G}_2^*/\mathfrak{G}_3^* \cong \mathfrak{F}_2/\mathfrak{R}^* \cdot \mathfrak{F}_3 \cong (\mathfrak{F}_2/\mathfrak{F}_3)/(\mathfrak{R}^* \cdot \mathfrak{F}_3/\mathfrak{F}_3).$$

$\mathfrak{F}_2/\mathfrak{F}_3$ is a free abelian group having as a basis the elements $[a_i, a_j]$ mod $\mathfrak{F}_3$, $i>j$, $i, j=1, 2, \cdots, n$. $\mathfrak{R}^* \cdot \mathfrak{F}_3/\mathfrak{F}_3$ is a subgroup of $\mathfrak{F}_2/\mathfrak{F}_3$ and is generated by elements $\prod_{j=1, j\neq i}^{n} [a_j, a_i]^{\mu_{ij}}$ mod $\mathfrak{F}_3$. Write the group $\mathfrak{F}_2/\mathfrak{F}_3$ additively, and replace $[a_j, a_i]$ mod $\mathfrak{F}_3$ by x_{ij}. Thus the corollary follows immediately.

COROLLARY 3. *If $L=L_1 \cup L_2$, then $[G, G]/[[G, G], G]$ is cyclic of order $|\mu_{12}|$.*

K. Reidemeister [7, p. 45] remarked that, for $L=L_1 \cup L_2$, $[a_1, a_2]$, taken as an element of G/G_3, is of order $|\mu_{12}|$. This result may be regarded as a corollary of Theorem 2.

5. **Examples.** Let $L=L_1 \cup L_2 \cup L_3$ be a link as given in Fig. 1, and G its group. The link L has three components, each pair of which has vanishing linking number. We shall therefore overlook the factor group G_2/G_3, which does not yield interesting invariants. In order to compute G_3/G_4, let F be the free group generated by a_{ij}, $i=1, 2, 3$; $j=1, 2, \cdots, k_i$, $k_1=k_2=2m$, $k_3=4m$. Write, for $j=1, 2, \cdots, m$,

$$\begin{aligned} b_{1\,2j-1} &= a_{3\,4j-3}, & b_{1\,2j} &= a_{3\,4j}^{-1}; \\ b_{2\,2j-1} &= a_{3\,4j-4}^{-1}, & b_{2\,2j} &= a_{3\,4j-1}; \\ b_{3\,4j-3} &= a_{1\,2j}, & b_{3\,4j-2} &= a_{2\,2j}; \\ b_{3\,4j-1} &= a_{1\,2j}^{-1}, & b_{3\,4j} &= a_{2\,2j+2}^{-1}. \end{aligned}$$

Write $v_{ij}=[b_{ij}, a_{ij}]$ and $r_{ij}=v_{ij}a_{ij}a_{i\,j+1}^{-1}$. Then G is presented by generators a_{ij} and relations r_{ij}, $i=1, 2, 3; j=1, 2, \cdots, k_i$. Define $a_i=a_{i1}$; $u_{i1}=e$, $u_{ij}=v_{i\,j-1}\,v_{i\,j-2}\cdots v_{i1}$, $j\neq 1$; $h_{ij}=u_{ij}a_i a_{ij}^{-1}$; $r_i=v_{i\,k_i}\,v_{i\,k_i-1}\cdots v_{i1}$. As given in the preceding section, G may be presented by generators a_{ij} and relations h_{ij} and r_i, $i=1, 2, 3; j=1, 2, \cdots, k_i$. Let $\mathfrak{G} \cong \{a_1, a_2, a_3 / \phi\psi(r_1), \phi\psi(r_2), \phi\psi(r_3)\}$. Then $G/G_4 \cong \mathfrak{G}/\mathfrak{G}_4$, which implies $G_3/G_4 \cong \mathfrak{G}_3/\mathfrak{G}_4$.

As before, $\mathfrak{F}$ denotes the free group generated by a_1, a_2, a_3. The following congruence identities may be verified straightforwardly: For any $u, u', v, w \in \mathfrak{F}$,

$$\begin{aligned} [uu', v, w] &\equiv [u, v, w][u', v, w] \bmod \mathfrak{F}_4, \\ [u^{-1}, v, w] &\equiv [u, v, w]^{-1} \bmod \mathfrak{F}_4, \\ [u^{-1}, v] &\equiv [v, u, u][u, v]^{-1} \bmod \mathfrak{F}_4. \end{aligned}$$

First we have

$$\begin{aligned}
\phi\psi(a_{1\ 2j-1}) &\equiv a_1 \bmod \mathfrak{F}_3,\\
\phi\psi(a_{1\ 2j}) &\equiv [a_3, a_1]a_1 \bmod \mathfrak{F}_3,\\
\phi\psi(a_{2\ 2j-1}) &\equiv a_2 \bmod \mathfrak{F}_3,\\
\phi\psi(a_{2\ 2j}) &\equiv [a_3, a_2]^{-1}a_2 \bmod \mathfrak{F}_3,\\
\phi\psi(a_{3\ 4j-3}) &\equiv a_3 \bmod \mathfrak{F}_3,\\
\phi\psi(a_{3\ 4j-2}) &\equiv [a_1, a_3]a_3 \bmod \mathfrak{F}_3,\\
\phi\psi(a_{3\ 4j-1}) &\equiv [a_2, a_3][a_1, a_3]a_3 \bmod \mathfrak{F}_3,\\
\phi\psi(a_{3\ 4j}) &\equiv [a_2, a_3]a_3 \bmod \mathfrak{F}_3.
\end{aligned}$$

Using the above identities, we have

$$\begin{aligned}
\phi\psi(v_{1\ 2j-1}) &= [\phi(a_{3\ 4j-3}), \phi(a_{1\ 2j-1})] \equiv [a_3, a_1] \bmod \mathfrak{F}_4,\\
\phi\psi(v_{1\ 2j}) &= [\phi(a_{3\ 4j}^{-1}), \phi(a_{1\ 2j})]\\
&\equiv [([a_2, a_3]a_3)^{-1}, [a_3, a_1]a_1] \bmod \mathfrak{F}_4\\
&\equiv [a_2, a_3, a_1]^{-1}[a_3, a_1]^{-1} \bmod \mathfrak{F}_4,\\
\phi\psi(v_{2\ 2j-1}) &\equiv [a_2, a_3, a_3][a_2, a_3, a_2]^{-1}[a_3, a_2]^{-1} \bmod \mathfrak{F}_4,\\
\phi\psi(v_{2\ 2j}) &\equiv [a_2, a_3, a_2][a_1, a_3, a_2][a_3, a_2, a_3][a_3, a_2] \bmod \mathfrak{F}_4,\\
\phi\psi(v_{3\ 4j-3}) &\equiv [a_3, a_1, a_3][a_1, a_3] \bmod \mathfrak{F}_4,\\
\phi\psi(v_{3\ 4j-2}) &\equiv [a_3, a_2, a_3]^{-1}[a_1, a_3, a_2]^{-1}[a_2, a_3] \bmod \mathfrak{F}_4,\\
\phi\psi(v_{3\ 4j-1}) &\equiv [a_3, a_1, a_3]^{-1}[a_2, a_3, a_1][a_1, a_3]^{-1} \bmod \mathfrak{F}_4,\\
\phi\psi(v_{3\ 4j}) &\equiv [a_3, a_2, a_3][a_2, a_3]^{-1} \bmod \mathfrak{F}_4,
\end{aligned}$$

for $j=1, 2, \cdots, m$. It follows that

$$\begin{aligned}
\phi\psi(r_1) &\equiv (\phi\psi(v_{1\ 2j}v_{1\ 2j-1}))^m \bmod \mathfrak{F}_4\\
&\equiv [a_2, a_3, a_1]^{-m} \bmod \mathfrak{F}_4,\\
\phi\psi(r_2) &\equiv (\phi\psi(v_{2\ 2j}v_{2\ 2j-1}))^m \bmod \mathfrak{F}_4\\
&\equiv [a_2, a_3, a_1]^m \bmod \mathfrak{F}_4,
\end{aligned}$$

and

$$\begin{aligned}
\phi\psi(r_3) &\equiv (\phi\psi(v_{2\ 4j}v_{3\ 4j-1}v_{3\ 4j-2}v_{3\ 4j-3}))^m \bmod \mathfrak{F}_4\\
&\equiv [a_2, a_3, a_1]^m[a_1, a_3, a_2]^{-m} \bmod \mathfrak{F}_4\\
&\equiv (\phi\psi(r_1))^{-1}(\phi\psi(r_2))^{-1} \bmod \mathfrak{F}_4.
\end{aligned}$$

Now $\mathfrak{G} \cong \{a_1, a_2, a_3 / \phi\psi(r_1), \phi\psi(r_2), \phi\psi(r_3)\}$, and

$$\mathfrak{G}/\mathfrak{G}_4 \cong \{a_1, a_2, a_3 / \phi\psi(r_1), \phi\psi(r_2), \phi\psi(r_3), \mathfrak{F}_4\}.$$

Thus, by Tietze operations,

$$\mathfrak{G}/\mathfrak{G}_4 \cong \{a_1, a_2, a_3 \,/\, [a_2, a_3, a_1]^m, [a_1, a_3, a_2]^m, \mathfrak{F}_d\}.$$

Define $\mathfrak{G}^* \cong \{a_1, a_2, a_3 \,/\, [a_2, a_3, a_1]^m, [a_1, a_3, a_2]^m\}$. Let $\mathfrak{N}^*$ be the normal

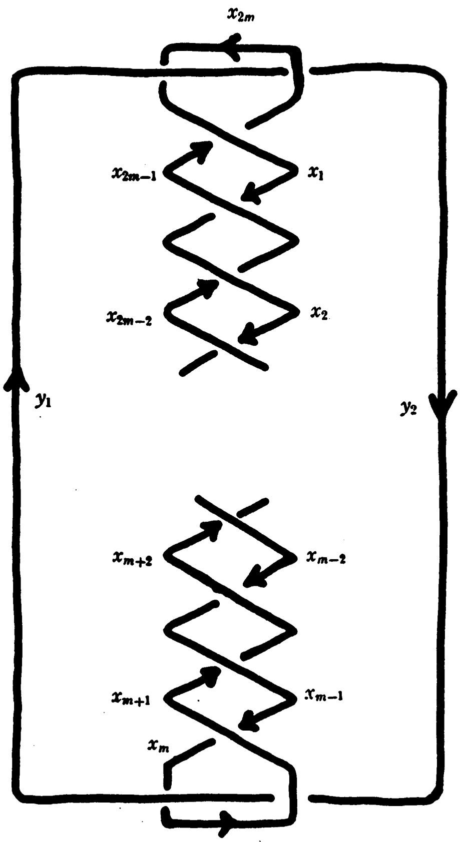

FIG. 2

subgroup generated by $[a_2, a_3, a_1]^m$ and $[a_1, a_3, a_2]^m$ in $\mathfrak{F}$. Then $\mathfrak{G}_3/\mathfrak{G}_4 \equiv \mathfrak{F}_3/\mathfrak{N}^* \cdot \mathfrak{F}_4 \equiv (\mathfrak{F}_3/\mathfrak{F}_4)/(\mathfrak{N}^* \cdot \mathfrak{F}_4/\mathfrak{F}_4)$. The group $\mathfrak{F}_3/\mathfrak{F}_4$ is free abelian of rank 8 [2; 4; 9]. We may choose as a basis for $\mathfrak{F}_3/\mathfrak{F}_4$ the elements $[a_1, a_2, a_1]$, $[a_1, a_2, a_2]$, $[a_1, a_2, a_3]$, $[a_1, a_3, a_1]$, $[a_1, a_3, a_2]$, $[a_1, a_3, a_3]$, $[a_2, a_3, a_2]$, $[a_2, a_3, a_3]$ mod $\mathfrak{F}_4$. The group $\mathfrak{N}^* \cdot \mathfrak{F}_4/\mathfrak{F}_4$ is free abelian with $[a_2, a_3, a_1]^m$, $[a_1, a_3, a_2]^m$ mod $\mathfrak{F}_4$ as basis. Hence G_3/G_4 is isomorphic to a direct product $J_m \times J_m \times B_6$ where J_m is a cyclic group of order m, and B_6 is a free abelian group of rank 6.

Consider another link $L = L_1 \cup L_2$ (Fig. 2) which has a vanishing linking number. In order to avoid the extensive use of double indices, we write the group G of L in the presentation:

$$G \cong \{x_1, x_2, \cdots, x_{2m}, y_1, y_2 / r_1, r_2, \cdots, r_{2m}, s_1, s_2\}$$

where

$$\begin{aligned} r_i &= [x_{2m-i}, x_i]x_i x_{i+1}^{-1}, && i \neq m, 2m, \\ r_m &= [y_1, x_m]x_m x_{m+1}^{-1}, \\ r_{2m} &= [y_1^{-1}, x_{2m}]x_{2m} x_1^{-1}, \\ s_1 &= [x_{2m}^{-1}, y_1]y_1 y_2^{-1}, \\ s_2 &= [x_{m+1}, y_2]y_2 y_1^{-1}. \end{aligned}$$

Let F be the free group generated by $x_1, x_2, \cdots, x_{2m}, y_1, y_2$, and denote $x = x_1$, $y = y_1$, $r = [y_1^{-1}, x_{2m}](\prod_{i=1}^{m-1}[x_i, x_{2m-i}])[y_1, x_m] \cdot (\prod_{i=m+1}^{2m-1}[x_i, x_{2m-i}])$, $s = [x_{m+1}, y_2][x_{2m}^{-1}, y_1]$. Define the substitutions ψ and ϕ such that $\psi(x_1) = x$, $\psi(x_{i+1}) = r_i x_{i+1} x_i^{-1}\psi(x_i)$, $i = 1, 2, \cdots, 2m-1$, $\psi(y_1) = y$, $\psi(y_2) = [x_{2m}^{-1}, y]y$, $\phi(x_i) = x$, $\phi(y_i) = y$. Then, according to Theorem 1,

$$G/G_5 \cong \{x, y / \phi\psi^2(r), \phi\psi^2(s), \mathfrak{F}_5\}$$

where $\mathfrak{F}$ is the free group generated by x and y. We are going to show that $\phi\psi^2(r) \equiv [x, y, x, y]^{m-1} \bmod \mathfrak{F}_5$ and $\phi\psi^2(s) \equiv [x, y, x, y]^{-m+1} \bmod \mathfrak{F}_5$, which will imply that

$$G/G_5 \cong \{x, y / [x, y, x, y]^{m-1}, \mathfrak{F}_5\}.$$

From $[u, av] = [u, a]a[u, v]a^{-1}$ it follows that, if $u \in \mathfrak{F}_s$, $v \in \mathfrak{F}_t$, $a \in \mathfrak{F}_q$, then $[u, av] \equiv [u, a][u, v] \bmod \mathfrak{F}_{s+t+q}$. The above congruence identity will be used frequently in this example.

It is immediate that $x = \phi\psi(x_1) = \phi\psi(x_2) = \cdots = \phi\psi(x_m)$ and $[y, x]x = \phi\psi(x_{m+1}) = \cdots = \phi\psi(x_{2m})$. It follows that, for $i = 1, 2, \cdots, m-1$,

$$\phi\psi([x_{2m-i}, x_i]) \equiv [[y, x]x, x] \equiv [y, x, x] \bmod \mathfrak{F}_4,$$

and, for $i = m+1, \cdots, 2m-1$,

$$\phi\psi([x_{2m-i}, x_i]) \equiv [y, x, x]^{-1} \bmod \mathfrak{F}_4.$$

By the definition of ψ,

$$\psi(x_m) = \left(\prod_{i=1}^{m-1}[x_{m+i}, x_{m-i}]\right)x$$

and

$$\psi(x_{2m}) = \left(\prod_{i=1}^{m-1} [x_i, x_{2m-i}]\right)[y, x_m]\psi(x_m).$$

Thus we have

$$\phi\psi^2(x_m) \equiv [y, x, x]^{m-1}x \bmod \mathfrak{F}_4$$

and

$$\phi\psi^2(x_{2m}) \equiv [y, x, x]^{-m+1}[y, \phi\psi(x_m)][y, x, x]^{m-1}x \equiv [y, x]x \bmod \mathfrak{F}_4.$$

Since $\psi([x_{2m-i}, x_i]) \in F_3$ and $\psi([y_1, x_m]) \in F_2$,

$$\psi(r) \equiv \psi([y_1^{-1}, x_{2m}])\left(\prod_{i=1}^{m-1} [x_i, x_{2m-i}]\right)\psi([y_1, x_m])\psi\left(\prod_{i=1}^{m-1} [x_i, x_{2m-i}]\right)^{-1}$$

$$\equiv \psi([y^{-1}, x_{2m}])\psi([y, x_m]) \bmod F_5.$$

Consequently

$$\begin{aligned}\phi\psi^2(r) &\equiv [y^{-1}, \phi\psi^2(x_{2m})][y, \phi\psi^2(x_m)]\\ &\equiv [y^{-1}, [y, x]x][y, [y, x, x]^{m-1}x]\\ &\equiv [x, y][x, y, x, y]^{m-1}[y, x] \equiv [x, y, x, y]^{m-1} \bmod \mathfrak{F}_5.\end{aligned}$$

On the other hand, $\psi(y_2) = [x_{2m}^{-1}, y]y$, $\phi\psi^2(y_2) = [\phi\psi(x_{2m}^{-1}), y]y \equiv [z^{-1}, y]y$ mod $\mathfrak{F}_4$, where $z = [y, x]x$. Moreover $\psi(x_{m+1}) = [y, x_m]\psi(x_m)$, and $\phi\psi^2(x_{m+1}) = [y, \phi\psi(x_m)]\phi\psi^2(x_m) \equiv [y, x][y, x, x]^{m-1}x \equiv [y, x, x]^{m-1}z$ mod $\mathfrak{F}_4$. Thus

$$\begin{aligned}\phi\psi^2(s) &\equiv [[y, x, x]^{m-1}z, [z^{-1}, y]y][z^{-1}, y]\\ &\equiv [[y, x, x]^{m-1}, y][z, [z^{-1}, y]y][z^{-1}, y]\\ &\equiv [y, x, x, y]^{m-1}[y, z^{-1}][z^{-1}, y]\\ &\equiv [x, y, x, y]^{-m+1} \bmod \mathfrak{F}_5.\end{aligned}$$

Therefore $G/G_5 \cong \{x, y / [x, y, x, y]^{m-1}, \mathfrak{F}_5\}$. The factor group $\mathfrak{F}_4/\mathfrak{F}_5$ is a free abelian group of rank 3. As its basis, we may choose $[x, y, x, x]$, $[x, y, x, y]$, $[x, y, y, y]$ mod $\mathfrak{F}_5$ [4]. The factor group G_4/G_5 is hence abelian and isomorphic with a direct product $J_{m-1} \times B_2$ where J_{m-1} is the cyclic group of order $m-1$, and B_2 is a free abelian group of rank 2, and the integer $m-1$ is a numerical invariant of the link.

Bibliography

1. R. Baer, *The higher commutator subgroups of a group*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 143–160.
2. K. T. Chen, *Integration in free groups*, Ann. of Math. vol. 54 (1951) pp. 147–162.
3. R. H. Fox, *Free differentiation*, forthcoming.

4. M. Hall, *A basis for free Lie ring and higher commutators in free groups*, Proceedings of the American Mathematical Society vol. 1 (1950) pp. 575–581.

5. W. Magnus, *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*, Math. Ann. vol. 111 (1935) pp. 259–280.

6. ———, *Über freie Faktorgruppen und freie Untergruppen gegebener Gruppen*, Monatshefte für Mathematik und Physik vol. 47 (1939) pp. 307–313.

7. K. Reidemeister, *Knotentheorie*, Berlin, Springer, 1932.

8. ———, *Einführung in die kombinatorische Topologie*, Braunschweig, Vieweg, 1932, pp. 46–49.

9. W. Witt, *Treue Darstellung der Lieschen Ringe*, J. Reine Angew. Math. vol. 177 (1937) pp. 152–160.

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