A BOUNDARY VALUE PROBLEM FOR A NONLINEAR DIFFERENTIAL EQUATION WITH A SMALL PARAMETER

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It is the purpose of this paper to prove Theorems 1 and 2 below which relate the existence, uniqueness, and general behavior of the solution, $y(x, \epsilon)$, for small $\epsilon > 0$, of the two-point boundary value problem

$$\epsilon y'' + f(x, y)y' + g(x, y) = 0,$$
$$y(0) = y_0, \qquad y(1) = y_1,$$

with the solution $u(x)$ of the corresponding "degenerate" initial value problem

$$f(x, u)u' + g(x, u) = 0, \qquad u(1) = y_1.$$

THEOREM 1 (*Existence*). *Let* $(0, y_0)$, $(1, y_1)$ *be two points in the real* (x, y)*-plane, and assume*:

(i) $f(x, y)$, $g(x, y)$ *are real functions such that the differential equation*

$$f(x, u)u' + g(x, u) = 0 \tag{1}$$

has a solution $u(x)$ *on* $0 \leqq x \leqq 1$, *with* $u(1) = y_1$ *and* $u(0) = u_0 \geqq y_0$.

(ii) $f(x, y)$, $g(x, y)$ *are of class* C' *in a region*

$$R: \qquad 0 \leqq x \leqq 1, \qquad |y - u(x)| \leqq a, \qquad a > 0,$$

which includes the point $(0, y_0)$.

(iii) *There exists a constant* $k > 0$ *such that* $f(x, y) \geqq k$ *for* (x, y) *in* R.

Then, for all sufficiently small $\epsilon > 0$, *there exists in* R *a solution* $y(x) = y(x, \epsilon)$ *of*

$$\epsilon y'' + f(x, y)y' + g(x, y) = 0 \tag{2}$$

satisfying the boundary conditions

$$y(0) = y_0, \qquad y(1) = y_1.$$

Further, $y(x, \epsilon) \to u(x)$, $y'(x, \epsilon) \to u'(x)$, *as* $\epsilon \to 0$, *uniformly on any subinterval* $0 < \delta \leqq x \leqq 1$.

REMARKS. From the proof of Theorem 1 it will be seen that the

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result of the theorem is valid if the hypotheses (ii) and (iii) are replaced by the somewhat weaker assumptions

(ii)* $f(x, y)$, $g(x, y)$ are of class C' in a region

$$R^*: \begin{array}{lr} 0 \leqq x \leqq \alpha < 1, \quad y_0 - \beta \leqq y \leqq u_0 + \beta & (\alpha, \beta > 0), \\ \alpha \leqq x \leqq 1, \quad |y - u(x)| \leqq \gamma & (\gamma > 0); \end{array}$$

(iii)* there exists a constant $k > 0$ such that $f(x, u(x)) \geqq k$, $0 \leqq x \leqq 1$, and

$$\int_y^{u_0} f(0, y)dy > 0, \qquad y_0 \leqq y < u_0;$$

and if R is replaced by R^*.

The assumption in (i) that $u_0 \geqq y_0$ is no restriction for if $u_0 < y_0$, then the change $y^* = -y$, $u^* = -u$ gives two equations (1)* and (2)* of the same type as (1) and (2) but with $u_0^* > y_0^*$. Also, by a change of variable of the type $x^* = px + q$, the interval $0 \leqq x \leqq 1$ can be replaced by an arbitrary bounded interval. It is further clear that if $\epsilon < 0$, a similar theorem will hold with the role of the left and right boundaries, $x = 0$ and $x = 1$, interchanged.

Assuming the existence of *both* $u(x)$ and $y(x, \epsilon)$, R. v. Mises [2][1] proved recently that as $\epsilon \to +0$, $y(x, \epsilon) \to u(x)$ and $y'(x, \epsilon) \to u'(x)$ uniformly on every subinterval $0 < \delta \leqq x \leqq 1$. He assumed that $f(x, y)$, $g(x, y)$ were continuous on a rectangle containing $(0, y_0)$ and $(x, u(x))$, and $f > 0$ there.

THEOREM 2 (*Uniqueness*). *Under the assumptions* (i), (ii), (iii) *of Theorem* 1, *for sufficiently small* $\epsilon > 0$, *there exists at most one solution* $y(x, \epsilon)$ *of* (2) *in* R_0 *satisfying the boundary conditions* $y(0) = y_0$, $y(1) = y_1$. *The region* R_0 *satisfies the same inequalities as* R *but with* a *replaced by a smaller quantity.*

REMARK. If (ii) and (iii) are replaced by (ii)* and (iii)* the result of Theorem 2 is valid in R_0^* where R_0^* satisfies the same inequalities as R^* but with β and γ replaced by smaller quantities.

PROOF OF THEOREM 1. *If* $y_0 < u_0$, let $y_0' > 0$. By virtue of (ii) there exists, on a sufficiently small interval to the right of $x = 0$, a solution of (2), $y(x) = y(x, \epsilon) = y(x, \epsilon; y_0, y_0')$, for which $y(0, \epsilon) = y_0$, $y'(0, \epsilon) = y_0'$, and such that $y(x, \epsilon)$ remains inside R. Put

$$\epsilon y_0' = \int_{y_0}^{u_0} f(0, y)dy + \mu, \tag{3}$$

[1] Numbers in brackets refer to the references cited at the end of the paper.

where ϵ is such that $\epsilon^{1/2}$ is less than the integral in (3), and μ is a real parameter whose magnitude is so small that $\epsilon y_0' > \epsilon^{1/2} > 0$. It will be shown that given any δ_1, $0 < \delta_1 < a$, then for small enough ϵ and μ there exists a $\xi > 0$ such that

$$\epsilon y'(\xi) = \epsilon^{1/2} \tag{4}$$

and if $y(\xi) = \eta$,

$$0 < \xi \leqq (u_0 + \delta_1 - y_0)\epsilon^{1/2}, \qquad |\eta - u_0| < \delta_1. \tag{5}$$

For small enough ϵ, and for $\mu > 0$, it will be the case that $\eta > u_0$, while for $\mu < 0$, $\eta < u_0$. In fact there exists a $\sigma > 0$ (independent of ϵ) such that for $\mu > 0$, $\eta - u_0 > \mu\sigma + O(\epsilon^{1/2})$ and for $\mu < 0$, $u_0 - \eta > -\mu\sigma + O(\epsilon^{1/2})$.

In order to prove these facts, let $y'(x) = \epsilon^{-1/2}$ for the first time at $x = \xi_1$, and $y(x) = u_0 + \delta_1$ for the first time at $x = \xi_2$, and define $\xi = \min(\xi_1, \xi_2)$. For small ϵ either ξ_1 or ξ_2 must exist and thus ξ exists. From the definition of ξ, and the mean value theorem for $y(x)$, it follows that $0 < \xi \leqq (u_0 + \delta_1 - y_0)\epsilon^{1/2}$. Integrating (2) from $x = 0$ to $x = \xi$, and using (3), we have if $\eta = y(\xi)$

$$\begin{aligned} \epsilon y'(\xi) = \int_\eta^{u_0} f(0, y)dy + \mu \\ + \int_0^\xi [f(0, y(x)) - f(x, y(x))]y'(x)dx - \int_0^\xi g(x, y(x))dx. \end{aligned}$$

From (ii) it follows that the last two terms are $O(\epsilon^{1/2})$, as $\epsilon \to 0$, and hence the above may be written as[2]

$$\epsilon y'(\xi) = \int_\eta^{u_0} f(0, y)dy + \mu + O(\epsilon^{1/2}). \tag{6}$$

For fixed $\delta_1 > 0$ and $|\mu|$ and ϵ small it follows from (6), since $y'(\xi) > 0$ and $f(0, u_0) > 0$, that $\eta = u_0 + \delta_1$ is impossible. Thus in fact $\xi = \xi_1$ and (4) follows. From (6) follows

$$\int_\eta^{u_0} f(0, y)dy + \mu = O(\epsilon^{1/2}). \tag{7}$$

Moreover (7) implies that if $|\mu|$ and ϵ are sufficiently small, $|\eta - u_0| < \delta_1$. This proves (5). The statements following (5) follow easily from (7) if $\sigma f(0, u_0) = 1/2$.

If $y_0 = u_0$, (3) becomes $\epsilon y_0' = \mu$, and if $\mu > \epsilon^{1/2} > 0$, define ξ_1, ξ_2, and

[2] In the following all O terms refer to $\epsilon \to 0$.

ξ as before. Then $0<\xi\leqq\delta_1\epsilon^{1/2}$, and (6) follows as before. If $\mu<-\epsilon^{1/2}\cdot<0$, define $x=\xi_1$ to be the first point such that $y'(x)=-\epsilon^{-1/2}$, and define $x=\xi_2$ to be the first point for which $y(x)=u_0-\delta_1$. If $\xi=\min(\xi_1, \xi_2)$ in this case, then $0<\xi\leqq\delta_1\epsilon^{1/2}$, and again (6) follows. The remainder of the proof of (4), (5), and the remarks following (5) is the same as for the case $y_0<u_0$.

Denote by $u(x;\xi,\eta)$ that solution of (1) for which $u(\xi;\xi,\eta)=\eta$. This solution will exist on $0\leqq x\leqq 1$ if ϵ and δ_1 are small enough since the given solution $u(x)$ is a continuous function of its initial point. Choose ϵ and δ_1 so small that this will be true. Then $u(x;\xi,\eta)\to u(x)$ uniformly on $0\leqq x\leqq 1$ as $(\xi,\eta)\to(0,u_0)$.

It will now be shown that as far to the right of ξ as $y(x,\epsilon)$, $y'(x,\epsilon)$ exist

$$|y(x,\epsilon)-u(x;\xi,\eta)| = O(\epsilon^{1/2}), \qquad \xi\leqq x\leqq 1, \tag{8}$$

$$|y'(x,\epsilon)-u'(x;\xi,\eta)| = O(\epsilon), \qquad \xi+\epsilon^{1/2}\leqq x\leqq 1. \tag{9}$$

By a familiar continuation argument, this will then prove the existence of $y(x,\epsilon)$ on $0\leqq x\leqq 1$.

In order to prove (8), let $z(x)=z(x,\epsilon;\xi,\eta)=y(x,\epsilon)-u(x;\xi,\eta)$. From (1) and (2), if $\xi\leqq x\leqq 1$, and $y(x,\epsilon)$ exists and is in R (if x is near enough to ξ, $y(x,\epsilon)$ certainly exists), then

$$\epsilon z''+f(x,y)z' = r, \tag{10}$$

where

$$\begin{aligned} r = {} & [g(x,u)-g(x,y)]+[f(x,u)-f(x,y)]u' \\ & + \epsilon f^{-2}(x,u)\left[f(x,u)\frac{\partial g}{\partial x}(x,u)-g(x,u)\frac{\partial f}{\partial x}(x,u)\right] \\ & + \epsilon f^{-3}(x,u)\left[\frac{\partial f}{\partial u}(x,u)g^2(x,u)-\frac{\partial g}{\partial u}(x,u)g(x,u)f(x,u)\right]. \end{aligned}$$

Because of the assumptions (ii), (iii), and the mean value theorem, it follows that there exists a constant $c>0$ such that

$$|r|\leqq c[|z|+\epsilon]. \tag{11}$$

Integrating (10) we obtain

$$\begin{aligned} z'(x) = {} & z'(\xi)\exp\left[-\frac{1}{\epsilon}\int_\xi^x f(t,y(t))dt\right] \\ & + \frac{1}{\epsilon}\int_\xi^x r\exp\left[-\frac{1}{\epsilon}\int_s^x f(t,y(t))dt\right]ds. \end{aligned} \tag{12}$$

Since $y'(\xi, \epsilon) = \epsilon^{-1/2}$, $|z'(\xi)| \leq 2\epsilon^{-1/2}$ if ϵ is sufficiently small, and if this is combined with the fact that $f(x, y) \geq k$, then (12) gives

$$
\begin{aligned}
|z'(x)| &\leq 2\epsilon^{-1/2} \exp\left[-\frac{k}{\epsilon}(x-\xi)\right] \\
&\quad + \frac{c\epsilon}{k} + \frac{c}{\epsilon}\int_\xi^x |z(s)| \exp\left[-\frac{k}{\epsilon}(x-s)\right] ds.
\end{aligned} \tag{13}
$$

Integrating the inequality (13) yields the following estimate for $|z(x)|$ (note that $z(\xi)=0$),

$$
\begin{aligned}
|z(x)| &\leq \frac{2\epsilon^{1/2}}{k} + \frac{c\epsilon}{k} \\
&\quad + \frac{c}{\epsilon}\int_\xi^x \left(\int_\xi^t |z(s)| \exp\left[-\frac{k}{\epsilon}(t-s)\right] ds\right) dt,
\end{aligned}
$$

and by an interchange of the order of integration in the last term we obtain

$$
|z(x)| \leq \frac{2\epsilon^{1/2}}{k} + \frac{c\epsilon}{k} + \frac{c}{k}\int_\xi^x |z(s)|\, ds.
$$

If ϵ is small enough, $c\epsilon < \epsilon^{1/2}$, and hence

$$
|z(x)| \leq \frac{3\epsilon^{1/2}}{k} + \frac{c}{k}\int_\xi^x |z(s)|\, ds.
$$

It follows easily from this that

$$
\int_\xi^x |z(s)|\, ds \leq \frac{3\epsilon^{1/2}}{c}\left[\exp\left(\frac{c}{k}(x-\xi)\right) - 1\right], \qquad \xi \leq x \leq 1,
$$

and therefore

$$
|z(x)| \leq \frac{3\epsilon^{1/2}}{k} \exp\left(\frac{c}{k}(x-\xi)\right), \qquad \xi \leq x \leq 1.
$$

This proves the estimate (8). Formula (9) follows directly from (8) and (13).

It still remains to show that y_0' can be chosen so that $y(1, \epsilon) = y_1$. From the remark after (5), if $\mu < 0$ and ϵ is small enough then $u_0 - \eta > -\mu\sigma/2$. Hence by the continuity of $u(x; \xi, \eta)$ in ξ and uniqueness, $u(x; \xi, \eta) < u(x)$, and on account of (8), $y(x,\epsilon) < u(x)$ on $\xi \leq x \leq 1$ if ϵ is sufficiently small. In particular $y(1, \epsilon) < y_1$. Similarly, for $\mu > 0$, and ϵ small enough, $y(1, \epsilon) > y_1$. Since $y(x, \epsilon)$ is in R for $0 \leq x \leq 1$

for small ϵ and μ, $y(x, \epsilon)$ is continuous in μ. By the continuity of $y(1, \epsilon)$ with respect to μ it follows that for ϵ sufficiently small, and some μ, $y(1, \epsilon)=y_1$. This, with (8) and (9), completes the proof of the existence theorem for the boundary value problem.

PROOF OF THEOREM 2. It will be shown that the solution $y(x, \epsilon) = y(x, \epsilon; y_0, y_0')$ considered as a function of y_0' satisfies

$$\frac{\partial y}{\partial y_0'}(1, \epsilon; y_0, y_0') > 0, \tag{14}$$

for ϵ sufficiently small. This clearly implies the uniqueness of $y(x, \epsilon)$ satisfying (2) and $y(0, \epsilon)=y_0$, $y(1, \epsilon)=y_1$.

Let

$$w(x) = w(x, \epsilon; y_0, y_0') = \frac{\partial y}{\partial y_0'}(x, \epsilon; y_0, y_0').$$

For fixed ϵ, $w(x)$, $w'(x)$, $w''(x)$ exist for $0 \leq x \leq 1$,[3] and from (2)

$$\epsilon w'' + fw' + \left(\frac{\partial f}{\partial y} y' + \frac{\partial g}{\partial y}\right) w = 0. \tag{15}$$

It is clear that

$$w(0) = 0, \qquad w'(0) = 1. \tag{16}$$

In order to prove (14) we first obtain appraisals for $w(x)$, $w'(x)$ near $x=0$. If (15) is integrated, the initial values (16) being used, then this yields

$$\epsilon w' + fw = \epsilon + \int_0^x \left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right) w dt. \tag{17}$$

Another integration results in the following expression for $w(x)$:

$$\begin{aligned} w(x) = &\int_0^x E(s, x; \epsilon) ds \\ &+ \frac{1}{\epsilon} \int_0^x \left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right) w \left[\int_r^x E(s, x; \epsilon) ds\right] dr, \end{aligned} \tag{18}$$

where

$$E(s, x; \epsilon) = \exp\left(-\frac{1}{\epsilon}\int_s^x f dt\right).$$

[3] This follows from an application of a well known theorem to (2); see, for example, [1].

By (ii), there exists a constant $m>0$ such that

$$\left|\frac{\partial f}{\partial x}(x, y(x)) - \frac{\partial g}{\partial y}(x, y(x))\right| \leqq m, \qquad 0 \leqq x \leqq 1, \tag{19}$$

and from (iii), $f(x, y(x)) \geqq k > 0$ on $0 \leqq x \leqq 1$. These facts, together with (18), imply

$$| w(x) | \leqq \frac{\epsilon}{k} + \frac{m}{k}\int_0^x | w(s) | ds, \tag{20}$$

and this in turn implies

$$\int_0^x | w(s) | ds \leqq \frac{\epsilon}{m}\left[\exp\left(\frac{mx}{k}\right) - 1\right]. \tag{21}$$

If (21) and (19) are applied to (18), there results

$$w(x) = \int_0^x E(s, x; \epsilon) ds + xO(\epsilon).$$

This can be written as

$$\begin{aligned} w(x) &= \frac{1}{F(x)}\int_0^x F(s)E(s, x; \epsilon) ds \\ &\quad + \frac{1}{F(x)}\int_0^x [F(x) - F(s)]E(s, x; \epsilon) ds + xO(\epsilon), \end{aligned} \tag{22}$$

where $F(x) = f(x, y(x))$.

We appraise the second integral on the right in (22). Let $\rho = \xi + \epsilon^{1/2}$, and suppose $x \geqq \rho + \epsilon^{1/2}$. By splitting the integral into two parts I_1 and I_2, where I_1 is the integral from 0 to ρ, I_2 the integral from ρ to x, we have first of all

$$I_1 \leqq 2M\int_0^\rho e^{-k/\epsilon^{1/2}} ds \leqq Me^{-k/\epsilon^{1/2}},$$

where $M = \max |f(x, y)|$ on R, and therefore $I_1/F(x) = O(\epsilon^2)$. Since both $x, s \geqq \xi + \epsilon^{1/2}$ in $\rho \leqq x \leqq 1$, it follows from (ii) and (9) that $F(x)$ satisfies a Lipschitz condition in $\rho \leqq x \leqq 1$. Thus

$$I_2/F(x) = O\left(\int_\rho^x (x - s)E(s, x; \epsilon) ds\right),$$

and by a simple calculation it is seen that this implies $I_2/F(x) = O(\epsilon^2)$.

Therefore, after integrating the first integral on the right of (22), we obtain

$$w(x) = \frac{\epsilon}{F(x)}\,[1 - E(0, x; \epsilon)] + O(\epsilon^2) + xO(\epsilon), \quad x \geqq \xi + 2\epsilon^{1/2}. \tag{23}$$

From (17) it now follows that

$$w'(x) = O(\epsilon + x), \qquad x \geqq \xi + 2\epsilon^{1/2}. \tag{24}$$

Suppose $\lambda < 0$, and $v(x)$ is defined by the relation

$$w(x) = v(x) \exp\left(\frac{\lambda x}{\epsilon}\right).$$

Clearly

$$w'(x) = \frac{\lambda}{\epsilon}\, w(x) + v'(x) \exp\left(\frac{\lambda x}{\epsilon}\right).$$

Then from (23) and (24), for small $x_1 \geqq \xi + 2\epsilon^{1/2}$,

$$v(x_1) > 0, \qquad v'(x_1) > 0. \tag{25}$$

Now (15) implies that $v(x)$ satisfies the equation

$$\epsilon v'' + (f + 2\lambda)v' + \left(\frac{\lambda^2}{\epsilon} + \frac{\lambda f}{\epsilon} + \frac{\partial f}{\partial y} y' + \frac{\partial g}{\partial y}\right) v = 0. \tag{26}$$

It follows from (9) that y' is bounded for $x_1 \leqq x \leqq 1$; and by (ii), the same is true for $\partial f/\partial y$, $\partial g/\partial y$. Thus there exists a constant $m^* > 0$ such that

$$\left|\frac{\partial f}{\partial y} y' + \frac{\partial g}{\partial y}\right| < m^*, \qquad x_1 \leqq x \leqq 1. \tag{27}$$

Let $x = x_2$, $x_1 < x_2 \leqq 1$, be the first point to the right of $x = x_1$ where $v'(x_2) = 0$. From (26) and (27) it is clear that $\epsilon^2 v''(x_2)$ has the same sign as $-(\lambda^2 + \lambda f)v(x_2)$ if ϵ is sufficiently small. But $v(x_2) > 0$ by (25), and $\lambda < 0$ can be chosen so that $\lambda^2 + \lambda f < 0$, since $f(x_2, y(x_2)) > 0$. Hence $v''(x_2) > 0$ if ϵ is sufficiently small. However this implies that $v'(x)$ is increasing at $x = x_2$, which contradicts the fact that $v'(x) > 0$, $x_1 \leqq x < x_2$, $v'(x_2) = 0$. Therefore $v'(x) > 0$ for $x_1 \leqq x \leqq 1$, and since $v(x_1) > 0$, it follows that $v(x) > 0$ for $x_1 \leqq x \leqq 1$. Thus $w(x) > 0$, $x_1 \leqq x \leqq 1$, and in particular $w(1) = \partial y/\partial y_0' > 0$ at $x = 1$. This proves (14), and hence the uniqueness theorem for μ sufficiently small.

For the case μ not small we note that the only use of Theorem 1 in the above proof was in the application of formula (9). If $y(x, \epsilon)$ is any solution of (2) remaining in R_0, then we show (9) must hold. There are two cases, either $|y_0'| \leqq \epsilon^{-1/2}$ or $|y_0'| > \epsilon^{-1/2}$. For the latter situation it is clear that $y(x, \epsilon)$ cannot remain in R_0 for small ϵ unless there exists a ξ, $0 < \xi < 1$, such that $|y'(x, \epsilon)| = \epsilon^{-1/2}$ for the first time at $x = \xi$. If ξ exists, then it follows by the mean value theorem that $\xi = O(\epsilon^{1/2})$. In either situation there exists a ξ for which $0 \leqq \xi < 1$, $\xi = O(\epsilon^{1/2})$, $|y'(\xi, \epsilon)| \leqq \epsilon^{-1/2}$. The existence of such a ξ is all that is needed to prove (8), (9) for the solution of (1), $u(x; \xi, \eta)$, which passes through ξ, $\eta = y(\xi, \epsilon)$ and which remains in R. But since $y(x, \epsilon)$ remains in R_0, it follows from (8) that $u(x; \xi, \eta)$ remains in R for ϵ sufficiently small. Therefore the argument used for μ small can be applied to extend the uniqueness to any solution $y(x, \epsilon)$ remaining in R_0.

For the more general case $\epsilon y'' + F(x, y, y') = 0$ the results of Theorems I and II are not valid unless F is severely restricted. Thus even $F = y' + (y')^3$ is not restricted enough for Theorem I to hold as can be seen by direct integration.

References

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