A THEOREM ON DIMENSION

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Every n-dimensional separable metric space can be homeomorphically imbedded in the closed $(2n+1)$-cube I^{2n+1}. This, of course, does not characterize n-dimensionality, for spaces of dimension greater than n can be imbedded homeomorphically in I^{2n+1}. A theorem due to Hurewicz[1] suggests that the existence of a more general kind of mapping into the n-cube I^n may characterize n-dimensionality for compact spaces. For compact spaces this theorem reduces to: If X, Y are compact and separable metric, $f:X\to Y$ is continuous, and $\dim X \geqq \dim Y$, then there exists $y \in Y$ such that $\dim f^{-1}(y) \geqq \dim X - \dim Y$. This suggests the possibility that if $\dim X = \dim Y$, one may be able to modify the map f slightly to obtain a map g, so that each $g^{-1}(y)$ has dimension zero, or is light. While this is not true in general, we show that it is possible if $\dim X = n$ and $Y = I^n$.

LEMMA 1. *Let P^n be an n-polyhedron and I^n be the n-cube, both with given simplicial decompositions. If $\psi:P^n\to I^n$ is simplicial and $\delta>0$, then there exists a map $\phi:P^n\to I^n$ such that:*

(1) *For any $x\in P^n$, $\|\phi(x)-\psi(x)\|<\delta$.*

(2) *For any $y\in I^n$, $\phi^{-1}(y)$ has at most one point in each (open) simplex of P^n.*

PROOF. Let $0<\eta<\min(\delta/2, \zeta/2)$ where ζ is the minimum diameter of the collection of n-simplexes of I^n. For each vertex q in the decomposition of I^n let $S^{n-1}(q) = \{y \mid y\in I^n, \|y-q\|=\eta\}$. On each $S^{n-1}(q)$ select $\aleph_0$ distinct points such that if q and q' are two vertices with $q'\in$ the closure of the set $\mathrm{St}(q)$, then no $(n-1)$-hyperplane determined by n of the selected points of $S^{n-1}(q)$ coincides with an $(n-1)$-hyperplane determined by n of the selected points of $S^{n-1}(q')$.

Let $\{p_i\}$ denote the collection of vertices of P^n. If $\psi^{-1}(q) = \bigcup p_{i_j}$, then let $\{\phi'(p_{i_j})\}$ be distinct points from the collection defined on $S^{n-1}(q)$. This defines ϕ' on all the vertices of P^n, and, if σ_l is an l-simplex, then ϕ' sends its vertices into $l+1$ independent points of I^n. Extend ϕ' linearly on each simplex of P^n to obtain a map $\phi:P^n\to I^n$. It can be seen that ϕ has the required properties.

DEFINITION. A map $f:X\to Y$ is said to be ϵ-light provided that: For any $y\in Y$, each component of $f^{-1}(y)$ has dia $<\epsilon$.

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[1] *Dimension theory*, Hurewicz and Wallman, Princeton, 1948, Theorem VI, 7, p. 91.

Given two spaces X, Y and a sequence $\{\epsilon_i\} \to 0$. Let F_{ϵ_i} denote the set of all ϵ_i-light maps of X into Y. Then $\bigcap_i F_{\epsilon_i} =$ all light maps $X \to Y$.

THEOREM. *Let X be a compact metric space. Then* dim $X \leqq n$ *if and only if there exists a light map of X into I^n.*

PROOF. *Necessity.* Let F denote the function space of maps $f: X \to I^n$ with the uniform topology, and denote by F_ϵ the set of ϵ-light elements of F.

(α) For any $\epsilon > 0$, F_ϵ is open in F.

Suppose that $f \in F_\epsilon$ and $\{h_i\} \to f$ with each $h_i \in F - F_\epsilon$. Then for each i there is a point $y_i \in I^n$ and a component c_i of $h_i^{-1}(y_i)$ such that dia $c_i \geqq \epsilon$. Let y be a limit point of $\{y_i\}$. Then $\{c_i\}$ contains a subsequence converging to a closed connected set C with dia $c \geqq \epsilon$. Furthermore $C \subset f^{-1}(y)$, contradicting the assumption that $f \in F_\epsilon$.

(β) For any $\epsilon > 0$, F_ϵ is dense in F.

Let $h \in F$ and $\delta > 0$. We must find $f \in F_\epsilon$ such that for any $x \in X$, $\|f(x) - h(x)\| < \delta$.

Take a simplicial decomposition of I^n into simplexes of dia $< \delta/2$. If $\{q_i\}$ are the vertices, then $\{h^{-1}(\text{St } (q_i))\}$ is a covering of X by open sets. Take a covering V of X such that

(1) V is a refinement of $\{h^{-1}(\text{St } (q_i))\}$,

(2) Each $v \in V$ has dia $< \epsilon$,

(3) Order $V \leqq n+1$.

Denote by $N(V)$ the nerve of V and by ζ a barycentric V-mapping[2] $\zeta: X \to N(V)$. Let $\{p_j\}$ be the collection of vertices of $N(V)$. Then the rule: $\psi'(p_j) =$ some q_i such that $h(v_j) \subset \text{St } (q_i)$ defines a simplicial map in $\psi': N(V) \to I^n$. By the lemma we can find a map $\psi: N(V) \to I^n$ satisfying conditions (1) (with $\delta/2$) and (2). The combined map $f = \psi\zeta$ has the property that $\|f(x) - h(x)\| < \delta$ for every $x \in X$; furthermore $f \in F_\epsilon$. For let C be a connected set in X such that $f(C)$ is a single point $y \in I^n$. Then $\zeta(C)$ is a connected set in $N(V)$, but it must be contained $\psi^{-1}(y)$ which is totally disconnected. Thus $\zeta(C)$ is a single point, and, since ζ is a barycentric V-mapping, this implies dia $C < \epsilon$.

(γ) The function space F is complete so that we may apply a theorem of Baire stating that the intersection of a countable number of open dense sets of a complete space has an intersection which is dense in the space. Thus if $\{\epsilon_i\} \to 0$, then the collection of light maps of X into $I^n (= \bigcap_i F_{\epsilon_i})$ is dense in F. This completes the proof of the necessity.

Sufficiency. This follows from the theorem of Hurewicz mentioned in the introduction.

[2] Op. cit. Definition V, 9, p. 69.

REMARK 1. The condition in the theorem that X be compact is important. For if X is the set of points in the plane which have at least one coordinate a rational number, then dim $X=1$, but there is no light closed map of X into the real line, and, a fortiori, none into the unit interval.

REMARK 2. It is clear that I^n could be replaced in the theorem by any n-manifold. This contrasts with the imbedding theorem, where there are n-dimensional sets that can be imbedded in manifolds of dimension less than $2n+1$, but cannot be imbedded in I^{2n}.

Added in proof. Our attention has been called to a paper by M. Katětov, *On rings of continuous functions*, Časopis pro Pěstování Matematiky a Fysiky vol. 75 (1950) pp. 1–16, Mathematical Reviews vol. 12 (1951) p. 119, which apparently contains several theorems closely related to our result, though not the same.

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