AN OPERATORIAL CHARACTERIZATION OF CERTAIN PARTITION POLYNOMIALS[1]

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Introduction. Let I_n $(n=1, 2, 3, \cdots)$ be the initial arithmetic interval consisting of the first n natural numbers $0, \cdots, n-1$. A sequence $(N_1, \cdots, N_x)$ of x sets of natural numbers is called an x-partition of I_n if I_n is the direct sum of the N's. I have shown elsewhere[2] that the number $f_n(x)$ of x-partitions of I_n satisfies the recursion relation

$$xf_n(x) = 1 + (x-1)\sum_{d|n} f_d(x), \tag{1}$$

the summation extending over all divisors d of n. In the above context x must, of course, be a positive natural number. It is clear, however, that (1) uniquely determines $f_n(x)$ as a polynomial in x with integer coefficients.

In this note we evaluate the partition polynomial $f_n(x)$ explicitly. This can easily be done by the usual combinatorial methods of multiplicative number theory, and we shall first record one expression for $f_n(x)$ obtained in this fashion. However, we are here mainly interested in the following operatorial characterization of $f_n(x)$. Let D be the operator of formal differentiation with respect to x acting on the ring R of polynomials in the indeterminate x over the field of rational numbers; and let F_k be the operator

$$F_k \equiv \frac{1}{k!} x^k D^k (x-1)^k \qquad (k \geqq 0) \tag{2}$$

($\equiv$ denotes operator identity). We shall show that the operators F_k commute among themselves and that

$$f_n(x) = F_{k_1} \cdots F_{k_m}[1] \tag{3}$$

(operands are enclosed in brackets) where

$$n = p_1^{k_1} \cdots p_m^{k_m} \tag{4}$$

is a prime factorization of n.

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[2] *Partitioning an arithmetic interval*, not yet published.

Combinatorial solution. Given $f_n(x)$ satisfying (1), let

$$\phi = \phi(x, s) = \sum_{n=1}^{\infty} f_n(x) n^{-s}$$

be the generating function of the f's. Then (1) may be written in the form

$$x\phi = \zeta + (x - 1)\zeta\phi$$

where $\zeta = \zeta(s)$ is the zeta function, so

$$\phi = \frac{1}{1 - (1 - \zeta^{-1})x} = \sum_{k=0}^{\infty} (1 - \zeta^{-1})^k x^k.$$

Separating out the coefficient of n^{-s} in this formula, we obtain the following explicit evaluation of $f_n(x)$:

$$f_n(x) = \sum_{k=0}^{\infty} c_k(n) x^k,$$

$$c_k(n) = \sum_{h=0}^{k} (-1)^h C_{k,h} \mu_h(n),$$

$$\mu_h(n) = (-1)^{k_1 + \cdots + k_m} C_{h,k_1} \cdots C_{h,k_m},$$

where n has the prime factorization (4); the $\mu_h(n)$'s are the generalized Moebius functions generated by ζ^{-h}. The partition polynomial $f_n(x)$ has degree $k_1 + \cdots + k_m$ and the highest power of x dividing it is max $(k_1, \cdots, k_m)$.

Operatorial solution. We define the operator L as follows:[3] it is to be a linear operator which operates on the basic polynomial operands x^i ($i \geqq 0$) of R with the result:

$$L[x^i] = \begin{cases} 0 & (i = 0), \\ x^{i-1} & (i \geqq 1). \end{cases} \tag{5}$$

Therefore

$$Lx \equiv 1, \tag{6}$$

so L is a left (but not a right) inverse of the operator x. However, since

[3] *Remark.* Each of the operators D and L can be expressed in terms of the other and the operator x in an infinite operator series as follows $D \equiv (1 + xL + x^2L^2 + \cdots)L$, $L \equiv (1 - (1/2!)xD + (1/3!)x^2D^2 - \cdots)D$. No question of convergence is involved here, for all except finitely many terms produce 0 when operating on an element of R.

$$x^h L^h[x^i] = \begin{cases} 0 & (i < h), \\ x^i & (i \geqq h), \end{cases} \tag{7}$$

L can under suitable conditions act like a right inverse of x. For example, it is clear from (7) that

$$D^k x^h L^h \equiv D^k x^{k-h} \qquad (h \leqq k). \tag{8}$$

Now consider the operators

$$E_k \equiv \frac{1}{k!} D^k x^k \qquad (k \geqq 0). \tag{9}$$

According to (8) we have

$$D^k \equiv D^k x^k L^k \equiv k! E_k L^k \qquad (k \geqq 0), \tag{10}$$

so

$$\begin{aligned} F_k &\equiv \frac{1}{k!} x^k D^k (x - 1)^k \\ &\equiv x^k E_k L^k (x - 1)^k \\ &\equiv x^k E_k (1 - L)^k \qquad (k \geqq 0) \end{aligned} \tag{11}$$

by repeated application of (6).

We now derive a relation on the E's which will induce a relation on the F's. From Leibnitz's formula or by induction we get

$$D^k x - x D^k \equiv k D^{k-1} \qquad (k \geqq 1),$$

whence by (8), (9), and (10)

$$\begin{aligned} k!(E_k - x E_k L) &\equiv D^k x^k - x D^k x^k L \\ &\equiv (D^k x - x D^k) x^{k-1} \\ &\equiv k D^{k-1} x^{k-1} \\ &\equiv k! E_{k-1} \qquad (k \geqq 1). \end{aligned}$$

Therefore according to (11)

$$\begin{aligned} F_k - x F_k L &\equiv x^k E_k (1 - L)^k - x^k x E_k L (1 - L)^k \\ &\equiv x^k (E_k - x E_k L)(1 - L)^k \\ &\equiv x^k E_{k-1} (1 - L)^k \\ &\equiv x F_{k-1} (1 - L) \qquad (k \geqq 1), \end{aligned}$$

which can be written in the form

$$(x-1)F_k \equiv x(F_k - F_{k-1})(1-L) \qquad (k \geqq 1). \tag{12}$$

Let us associate operators F^* to the operators F as follows:

$$\begin{aligned} F_0^* &\equiv F_0(\equiv 1), \\ F_k^* &\equiv F_k - F_{k-1} \qquad (k \geqq 1), \end{aligned} \tag{13}$$

whereupon

$$F_k \equiv \sum_{h=0}^{k} F_h^* \qquad (k \geqq 0). \tag{14}$$

By a complex we shall mean an operator C formed by composition of F's:

$$C \equiv F_{k_1} \cdots F_{k_m} \tag{15}$$

and we put

$$C^* \equiv F_{k_1}^* \cdots F_{k_m}^*, \tag{16}$$

so that for any two complexes A and B we have

$$(AB)^* \equiv A^*B^*. \tag{17}$$

The number of positive k's occurring in the expression (15) for the complex C will be called the length of (that expression of) C. Clearly the identity operator 1 is the only complex of length 0.

We now show that

$$(x-1)C \equiv xC^*(1-L) \qquad (\text{length } C \geqq 1) \tag{18}$$

for any complex C of positive length by induction on the length l of C. Formula (12) together with definition (13) assures us that (18) is valid for any complex C of length $l=1$. Assume (18) true for any complex of positive length less than l with $l>1$: we are to prove (18) for C of length l. Since $l>1$, C can be factored into two complexes A and B each of positive length less than l so that $C=AB$. Therefore using (6) and (17) and the fact that both A and B satisfy (18) we have

$$\begin{aligned} (x-1)AB &\equiv xA^*(1-L)B \\ &\equiv xA^*B - xA^*L(xB - xB^*(1-L)) \\ &\equiv xA^*B - xA^*B + xA^*B^*(1-L) \\ &\equiv x(AB)^*(1-L). \end{aligned}$$

This shows that C satisfies (18), so by induction (18) is valid for all complexes C of positive length.

We are now in a position to verify that the operators F commute with one another over R. It is clear from the rearranged form

$$C = x(C - C^*) + xC^*L$$

of formula (18), applied to the complex $C = F_h F_k$, that the commutativity of the F's follows by induction both on $h+k$ and on the degree of the polynomial operand in virtue of the degree reducing property of the operator L.[4]

Let us finally consider the operator

$$S \equiv x \sum_{h_1=0}^{k_1} \cdots \sum_{h_m=0}^{k_m} (F_{h_1} \cdots F_{h_m})^*(1 - L).$$

We see from (16) and (14) that

$$S \equiv xF_{k_1} \cdots F_{k_m}(1 - L). \tag{19}$$

On the other hand, formula (18) applies to every term of the sum S except the single term of length 0, wherein all h's are 0, so

$$S - x(1 - L) \equiv (x - 1) \sum_{h_1=0}^{k_1} \cdots \sum_{h_m=0}^{k_m} F_{h_1} \cdots F_{h_m} - (x - 1). \tag{20}$$

By eliminating S between (19) and (20) we obtain the following relation for complexes:

$$\begin{aligned} xF_{k_1} \cdots F_{k_m} \equiv 1 + (x - 1) \sum_{h_1=0}^{k_1} \cdots \sum_{h_m=0}^{k_m} F_{h_1} \cdots F_{h_m} \\ + x(F_{k_1} \cdots F_{k_m} - 1)L. \end{aligned}$$

Since $L[1] = 0$, we conclude from this relation that the functions $f_n(x)$ defined by (3) and (4) satisfy (1)—as was to be proved.

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[4] It can be shown more generally and independently of the particular nature of the ring R of operands that the operators $F_{hk} \equiv a^h b^k D^k D^h b^h a^k$ $(h, k \geqq 0)$ commute among themselves, where a and b are polynomials in x of at most first degree.