A UNIQUENESS THEOREM FOR A CLASS OF HARMONIC FUNCTIONS

A. J. LOHWATER[1]

In this note we shall establish a uniqueness theorem for the class of functions $u(r, \theta)$, harmonic in $|z| < 1$, for which

$$(1) \qquad \int_0^{2\pi} |u(r, \theta)| \, d\theta < M, \qquad z = re^{i\theta},$$

where M is a finite constant independent of r.

THEOREM. *Let* $u(r, \theta)$ *be harmonic in* $|z| < 1$ *and there satisfy* (1). *Let* $\lim_{r\to 1} u(r, \theta) = 0$ *for almost all* θ *in* $0 \leqq \theta \leqq 2\pi$, *and let* $\lim_{r\to 1} u(r, \theta) = \pm \infty$ *for all* θ *belonging to a countable set* E *in* $0 \leqq \theta \leqq 2\pi$. *If* $\lim_{r\to 1} u(r, \theta)$, *wherever else it may exist, is not infinite, then there exist real constants* c_n *with* $\sum_{n=1}^{\infty} |c_n| < \infty$ *such that, in* $|z| < 1$,

$$(2) \qquad u(r, \theta) \equiv \sum_{n=1}^{\infty} c_n K(r, \theta - \theta_n),$$

where $\bigcup_{n=1}^{\infty} \theta_n = E$ *and* $K(r, \theta - \alpha) = (1 - r^2)/(1 + r^2 - 2r \cos(\theta - \alpha))$.

A harmonic function satisfying (1) in $|z| < 1$ has an integral representation

$$(3) \qquad u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} K(r, \theta - \phi) d\mu(\phi),$$

where $\mu(\phi)$ is of bounded variation in the interval $0 \leqq \phi \leqq 2\pi$. We consider the Lebesgue decomposition of $\mu(\phi)$:

$$(4) \qquad \mu(\phi) = \nu(\phi) + g(\phi),$$

where $\nu(\phi)$ is absolutely continuous with $\mu'(\phi) = \nu'(\phi)$ almost everywhere, and $g(\phi)$ is of bounded variation with $g'(\phi) = 0$ almost everywhere. Now, for any ϕ for which $\mu'(\phi)$ exists, which is the case almost everywhere, $\lim_{r\to 1} u(r, \phi) = \mu'(\phi)$. Since $\lim_{r\to 1} u(r, \theta) = 0$ almost everywhere, we have $\nu'(\phi) = 0$ almost everywhere, so that $\nu(\phi)$ in (4) is identically constant. Now, it is known[2] that, for any point of discontinuity θ_0 of $\mu(\phi)$, $\lim_{r\to 1} u(r, \theta_0) = \pm \infty$. Consequently the points

Presented to the Society, September 7, 1951; received by the editors July 9, 1951.

[1] The work on this paper was performed under sponsorship of ONR.

[2] Cf. the author's paper *The boundary values of a class of meromorphic functions*, to appear in Duke Math. J.

of discontinuity of $\mu(\phi)$ are contained in the set E. If we write $g(\phi) = g_1(\phi) + \psi(\phi)$, where $g_1(\phi)$ is a continuous function of bounded variation with $g_1'(\phi) = 0$ almost everywhere and $\psi(\phi)$ is a step function, it follows from [1, pp. 127–128] that, unless $g_1(\phi)$ reduces to a constant, $g'(\phi)$ is infinite on a noncountable set of points. This implies that $\lim_{r\to 1} u(r, \theta)$ is infinite on a noncountable set of values of θ, which is contrary to hypothesis. Hence $g_1(\phi)$ is identically constant, and $\mu(\phi)$ reduces to a pure step function.

Since $\mu(\phi)$ in (3) reduces to a step function, we may replace the Stieltjes integral there by a series; more precisely: there exists a sequence of real numbers $\{c_n\}$ $(n = 1, 2, \cdots)$ with $\sum_{n=1}^{\infty} |c_n| < \infty$ such that

$$u(r, \theta) = \sum_{n=1}^{\infty} c_n K(r, \theta - \theta_n),$$

and where $2\pi c_n$ represents the saltus of the step function $\psi(\phi)$ at the jump points θ_n. Hence the theorem is proved.

The following corollary is an easy consequence of our theorem: *Let $u(r, \theta)$ be harmonic in $|z| < 1$ and there satisfy* (1); *let* $\lim_{r\to 1} u(r, \theta) = 0$ *for almost all θ in $0 \leqq \theta \leqq 2\pi$ and let* $\lim_{r\to 1} u(r, \theta)$, *wherever else it may exist, not be infinite. Then $u(r, \theta)$ is identically zero in* $|z| < 1$.[3] Indeed, since $\lim_{r\to 1} u(r, \theta)$, wherever it exists, cannot be infinite, it follows that all the constants c_n in (2) must be zero.

To show that condition (1) is essential we exhibit the function

$$u(r, \theta) = \frac{\partial}{\partial\theta}\left\{\frac{1 - r^2}{1 + r^2 - 2r\cos\theta}\right\} = \frac{-2r(1 - r^2)\sin\theta}{[1 + r^2 - 2r\cos\theta]^2}$$

which is harmonic in $|z| < 1$ and has the property that, for all θ, $\lim_{r\to 1} u(r, \theta) = 0$.

Bibliography

1. S. Saks, *Theory of the integral*, Warsaw, 1937.

The University of Michigan

[3] We remark that P. C. Rosenbloom has proved a weaker form of this corollary assuming that $\lim_{r\to 1} u(r, \theta) = 0$ for *every* value of θ. His result will appear in his forthcoming book on partial differential equations.