ELEMENTARY PROOF OF A NORM THEOREM

HARLEY FLANDERS

The result below was first obtained as a consequence of the theory of cyclic algebras [1].[1] The proof given here uses only the Hilbert norm theorem [2].

Let K/k be a cyclic extension of degree mn with generating automorphism σ. Let $K \supset F \supset k$ with $[K:F]=m$, $[F:k]=n$ so that F is the fixed field of the subgroup $\langle \sigma^n \rangle$ of the cyclic group $\langle \sigma \rangle$. Let $a \in k$, $a \neq 0$, and $a^m = N_{K/k}A$. Then $a = N_{F/k}B$ for some $B \in F$.

To prove this, set $E = AA^{\sigma} \cdots A^{\sigma^{n-1}}$. Then $N_{K/F}E = (N_{K/F}A) \cdot (N_{K/F}A)^{\sigma} \cdots (N_{K/F}A)^{\sigma^{n-1}} = N_{F/k}N_{K/F}A = a^m$, hence $N_{K/F}(aE^{-1}) = 1$. By the Hilbert norm theorem, $aE^{-1} = C/C^{\sigma^n}$ for a non-zero element C of K. Hence $C^{\sigma^n} = a^{-1}EC$ and $C^{\sigma^{n+1}} = a^{-1}E^{\sigma}C^{\sigma}$. We set $B = AC/C^{\sigma}$ and have

$$B^{\sigma^n} = \frac{A^{\sigma^n}C^{\sigma^n}}{C^{\sigma^{n+1}}} = \frac{A^{\sigma^n}EC}{E^{\sigma}C^{\sigma}} = \frac{AC}{C^{\sigma}} = B,$$

hence $B \in F$. Finally,

$$N_{F/k}B = AA^{\sigma} \cdots A^{\sigma^{n-1}} \frac{CC^{\sigma} \cdots C^{\sigma^{n-1}}}{C^{\sigma}C^{\sigma^2} \cdots C^{\sigma^n}} = E\frac{C}{C^{\sigma^n}} = a.$$

Professor Jacobson pointed out to the author that the same argument can be used to derive Theorem 28 on page 47 of [3], starting from Theorem 27, which is a noncommutative form of the Hilbert norm theorem.

References

1. A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Colloquium Publications, vol. 24, 1939, Chap. 7, Theorem 16.

2. ———, *Modern higher algebra*, Chicago, 1937, Chap. 9, Theorem 5.

3. N. Jacobson, *The theory of rings*, New York, 1943.

California Institute of Technology

Received by the editors July 23, 1951.

[1] Numbers in brackets refer to the references at the end of the paper.