A REMARK ON THE $\alpha+\beta$ THEOREM

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In [1],[1] Mr. Benjamin Lepson proved the following theorem: Given $\alpha>0, \beta>0$, whenever $1\geqq\alpha+\beta$, there exist sets A and B of positive integers such that the Schnirelmann densities of A, B, and $A+B$ are α, β, and $\alpha+\beta$ respectively. In this note we establish the following stronger result:

THEOREM. *Given $\alpha\geqq 0$, $\beta\geqq 0$, $\gamma\geqq 0$, whenever $1\geqq\gamma\geqq\alpha+\beta$, there exist sets A and B of positive integers such that $d(A)=\alpha$, $d(B)=\beta$, and $d(A+B)=\gamma$. The symbol $d(A)$ denotes the Schnirelmann density of A.*

PROOF. Let $\gamma-\alpha=\delta$, $\gamma-\beta=\epsilon$. Then $\delta\geqq\beta$, $\epsilon\geqq\alpha$. For $n\geqq 1$, define integers a_n, b_n such that when n is even

$$\alpha(n+1)!-\alpha n!\leqq a_n<\alpha(n+1)!-\alpha n!+1,$$
$$\delta(n+1)!-\delta n!\leqq b_n<\delta(n+1)!-\delta n!+1,$$

and when n is odd

$$\epsilon(n+1)!-\epsilon n!\leqq a_n<\epsilon(n+1)!-\epsilon n!+1,$$
$$\beta(n+1)!-\beta n!\leqq b_n<\beta(n+1)!-\beta n!+1.$$

Thus in general

$$\alpha(n+1)!-\alpha n!\leqq a_n\leqq(n+1)!-n!,$$
$$\beta(n+1)!-\beta n!\leqq b_n\leqq(n+1)!-n!,$$

and

$$\gamma(n+1)!-\gamma n!\leqq a_n+b_n<\gamma(n+1)!-\gamma n!+2.$$

Moreover, $\liminf a_n/(n+1)!=\alpha$, $\liminf b_n/(n+1)!=\beta$.

Construct A to be the set consisting of the number 1 and all numbers $n!+1, n!+2, \cdots, n!+a_n$ for all integers $n\geqq 1$. Construct B in a similar way using b_n in place of a_n. Since

$$a_{n-1}+(n-1)!\geqq A(n!)=1+a_1+a_2+\cdots+a_{n-1}$$
$$\geqq 1+\alpha\sum_{j=1}^{n-1}((j+1)!-j!)=1-\alpha+\alpha n!\geqq\alpha n!,$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

we have

$$\frac{a_{n-1}}{n!} + \frac{1}{n} \geqq \frac{A(n!)}{n!} \geqq \alpha.$$

The lower limit of the left member as n goes to infinity is α, therefore

$$\liminf \frac{A(n!)}{n!} = \alpha, \quad \text{so that} \quad d(A) \leqq \alpha.$$

Let x be any number, and let n be the number such that $(n-1)! < x \leqq n!$. If x is in A, then $(n-1)! < x \leqq (n-1)! + a_{n-1}$, so

$$\frac{A(x)}{x} = \frac{A((n-1)!) + x - (n-1)!}{(n-1)! + x - (n-1)!} \geqq \frac{A((n-1)!)}{(n-1)!}.$$

If x is not in A, then $(n-1)! + a_{n-1} < x \leqq n!$, and we have

$$\frac{A(x)}{x} = \frac{A(n!)}{x} \geqq \frac{A(n!)}{n!}.$$

Together with the fact that $A(n!)/n! \geqq \alpha$ for all n, we have $d(A) \geqq \alpha$, and hence $d(A) = \alpha$. Similarly $d(B) = \beta$.

Let $C = A + B$, so that C consists of all numbers a, b, and $a+b$ with a in A and b in B. If a is in A and $a \leqq n!$, then $a \leqq (n-1)! + a_{n-1}$. If b is in B and $b \leqq n!$, then $b \leqq (n-1)! + b_{n-1}$. This implies that $C(n!) \leqq 2(n-1)! + a_{n-1} + b_{n-1}$. Divide by $n!$ and take the limit, and since $a_{n-1} + b_{n-1} < \gamma n! - \gamma(n-1)! + 2$, we have $\liminf C(n!)/n! \leqq \gamma$. Hence $d(C) \leqq \liminf C(x)/x \leqq \gamma$. Let x be any positive integer. Either $C(x) = x \geqq \gamma x$ or $C(x) < x$. In the latter case, by Mann's strong proposition [2 or 3] there exists an integer $z \leqq x$, z not in C, such that

$$\frac{C(x)}{x} \geqq \frac{A(z) + B(z)}{z}.$$

Now if y is defined by

$$(y-1)! < z \leqq y!,$$

we have since z is not in A or in B,

$$\frac{A(z) + B(z)}{z} \geqq \frac{A(y!) + B(y!)}{y!}.$$

Since

$$A(y!) + B(y!) = 1 + \sum_{j=1}^{y-1} a_j + 1 + \sum_{j=1}^{y-1} b_j = 2 + \sum_{j=1}^{y-1} (a_j + b_j)$$
$$\geqq 2 + \gamma \sum_{j=1}^{y-1} [(j+1)! - j!] = 2 + \gamma y! - \gamma > \gamma y!,$$

we have $C(x)/x \geqq \gamma$. Then $d(C) \geqq \gamma$ so that $d(C) = \gamma$, and the theorem is proved.

It should be noted that the theorem will still be true if we replace the Schnirelmann density by the asymptotic density, which of A, for example, is defined to be lim inf $A(x)/x$ as x goes to infinity. Each of the sets A, B, C of the above proof has the same asymptotic density as Schnirelmann density.

REFERENCES

1. B. Lepson, *Certain best possible results in the theory of Schnirelmann density*, Proceedings of the American Mathematical Society vol. 1 (1950) pp. 592–594.

2. H. B. Mann, *A proof of the fundamental theorem on the density of sums of positive integers*, Ann. of Math. vol. 43 (1942) pp. 523–527.

3. E. Artin and P. Scherk, *On the sum of two sets of integers*, Ann. of Math. vol. 44 (1943) pp. 138–142.

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