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ON SOME FUNCTIONS HOLOMORPHIC IN AN INFINITE REGION

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S. Mandelbrojt indicated the following proposition: If a function is holomorphic and bounded in a half-strip of the z-plane containing the half-axis ox as a part of its central line and if this function and a certain infinite sequence of its derivatives vanish at the origin, then it is identically zero. The proof of this proposition is based upon a result of Mandelbrojt [1, p. 372].[2] In the present paper, we consider a function $F(z)$ holomorphic in a region Δ of the z-plane defined by $x \geqq d$, $|y| \leqq g(x)$, where $-\infty < d < 0$ and where $g(x)$ is a certain positive continuous function tending to zero with $1/x$. In this case if, in Δ, $F(z)$ tends to zero rapidly enough and uniformly with respect to y as x tends to infinity, and if $F(z)$ and a certain infinite sequence of its derivatives vanish at the origin, then $F(z)$ is identically zero. In order to establish our proposition, we prove at first a lemma by means of the following theorem of G. Valiron [3, p. 62, §32]:

THEOREM V. *Let $Y(X)$ be a real function having a first derivative for $X \geqq X_0$ such that*

$$\lim_{x=\infty} \frac{XY'(X)}{\psi(X)} = 1; \qquad \psi(X) \geqq 1, \quad X \geqq X_0; \qquad \lim_{x=\infty} \frac{X\psi'(X)}{[\psi(X)]^2} = 0.$$

Let $\Phi(X)$ be an entire function and let $M(r) = \max_{|z|=r} |\Phi(z)|$. Then a necessary and sufficient condition that

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[2] Numbers in brackets refer to the bibliography at the end of this paper.

$$\log M(r) \sim e^{Y(X)}, \qquad X = \log r,$$

is that

$$\nu(r) \sim Y'(X)e^{Y(X)} \sim \frac{d}{dX} \log M(r),$$

where $\nu(r)$ *is the rank of the maximum term of the highest rank of the Taylor expansion of* $\Phi(z)$ *corresponding to the value* $|z| = r$.

LEMMA. *Let* $\Phi(z) = \sum_0^\infty \phi(n)z^n$ *and let* $\mu(r)$ *be the value of the maximum terms of* $|\phi(n)|r^n$ $(n=0, 1, 2, \cdots)$. *If*[3]

$$\mu(r) \sim K[(\log_2 r)(\log_3 r) \cdots (\log_{p+1} r)]^{\log r} \qquad (K = \text{const.} > 0),$$

then for any given $\epsilon > 0$ $(\epsilon < 1)$, *we have, for* n *sufficiently large,*

$$|\phi(n)| < \exp\{-\exp[\omega_p(e^{(1-\epsilon)n})]\}$$

and, for a sequence $\{n_k\}$ *such that* n_{k+1}/n_k *tends to* 1 *as* k *tends to infinity,*

$$|\phi(n_k)| > \exp\{-n_k \exp[\omega_p(e^{(1+\epsilon)n_k})]\}$$

where p *is a positive integer and where* $\xi = \omega_p(\eta)$ *is the inverse function of* $\eta = \xi(\log \xi)(\log_2 \xi) \cdots (\log_{p-1} \xi)$.

PROOF. Since [**3**, p. 111 and **4**, p. 32, chap. II]

$$\log M(r) \sim \log \mu(r) \sim (\log r)(\log_3 r + \log_4 r + \cdots + \log_{p+2} r),$$

we have, by Theorem V,

$$\nu(r) \sim \log [(\log_2 r)(\log_3 r) \cdots (\log_{p+1} r)].$$

Considering with Valiron a polygon of Newton and using his notations, we see that

$$n \sim \log [(\log_2 R_n)(\log_3 R_n) \cdots (\log_{p+1} R_n)].$$

$\omega_p(\eta)$ being an increasing function, it follows that

$$\exp\{\exp[\omega_p(e^{(1-\epsilon)n})]\} < e^{G_n} = e^{G_0}R_1R_2 \cdots R_n$$
$$< \exp\{n \exp[\omega_p(e^{(1+\epsilon)n})]\}$$

for n sufficiently large. The lemma will then be completely established by Valiron's reasonings.

The following result is an immediate corollary of our lemma:

[3] We write $\log_0 x = x$ and $\log_k (x) = \log (\log_{k-1} x)$, k being a positive integer and x being sufficiently large.

COROLLARY. *If for a given* $\epsilon > 0$,

$$\phi(n) = \exp \{ -n \exp [\omega_p(e^{(1+\epsilon)n})] \}$$

for n sufficiently large, then we have

$$\mu(r) \leqq [(\log_2 r)(\log_3 r) \cdots (\log_{p+1} r)]^{\log r}$$

for r sufficiently large.

Now we can prove our main theorem:

THEOREM. *Let* $g(x)$ *be a positive continuous function defined for* $x \geqq d$ $(-\infty < d < 0)$ *decreasing to zero with* $1/x$ *for x sufficiently large and satisfying*

$$g(x) = O[g(x + \eta)] \qquad (x \to \infty) \tag{1}$$

for $|\eta|$ *sufficiently small. Denote by* Δ *the region of the z-plane defined by* $x \geqq d$, $|y| \leqq g(x)$.

Let $\{\nu_n\}$ *and* $\{q_n\}$ *be two complementary sequences of non-negative integers* [1] *such that the upper density function* [1] $D^*(q)$ *of* $\{q_n\}$ *satisfies, for q sufficiently large,*

$$D^*(q) < \frac{b}{(\log q)(\log_2 q) \cdots (\log_{p+1} q)} \qquad \left(0 < b = \text{const.} < \frac{1}{2}\right). \tag{2}$$

Suppose that $F(z)$ *is a function holomorphic in* Δ *and satisfying*

$$F^{(\nu_n)}(0) = 0 \tag{3}$$

and, for a given $\epsilon > 0$,

$$F(z) = O\{[g(x)]^{\exp \omega_p\{[g(x)]^{-1-\epsilon}\}}\} \qquad (z \text{ in } \Delta,\ x \to \infty). \tag{4}$$

Then we conclude $F(z) \equiv 0$.

PROOF. We can evaluate the moduli of all the derivatives of $F(z)$ on the half-axis ox: $x \geqq 0$, $y = 0$. Let us put

$$h(x) = \min [x - d, g(\xi)] \qquad [x \geqq 0,\ |x - \xi| \leqq g(x)]$$

and construct in the z-plane circles $C(x)$: $|z - x| \leqq h(x)$ which are evidently situated in Δ. We have

$$F^{(n)}(x) = \frac{n!}{2\pi i} \int_{C(x)} \frac{F(z)}{(z - x)^{n+1}} dz \qquad (x \geqq 0).$$

By hypotheses there exist positive constants A, B, E and $x_0 > d$ such that:

$|F(z)| \leqq A$ for $z \in \Delta \cap \{\Re(z) \leqq x_0 + g(x_0)\}$;

$|F(z)| \leqq B[g(x)]^{\exp \omega_p\{[g(x)]^{-1-\epsilon}\}}$ for $z \in \Delta \cap \{\Re(z) \geqq x_0 - g(x_0)\}$;

$g(x)$ decreases for $x \geqq x_0 - g(x_0)$;

$h(x) \geqq E$ for $x \leqq x_0 + g(x_0)$.

It follows that

$$|F^{(n)}(x)| \leqq A \cdot \frac{n!}{E^n} \quad \text{for} \quad 0 \leqq x \leqq x_0;$$

$$\begin{aligned} |F^{(n)}(x)| &\leqq Bn! \frac{[g(x - g(x))]^{\exp \omega_p\{[g(x-g(x))]^{-1-\epsilon}\}}}{[h(x)]^n} \\ &\leqq Bn! \frac{[g(x - g(x))]^{\exp \omega_p\{[g(x-g(x))]^{-1-\epsilon}\}}}{[g(x + g(x))]^n} \\ &= Bn!\Omega_n(x, g(x)), \text{ say, for } x \geqq x_0. \end{aligned}$$

We are going to find an upper bound of $\Omega_n(x, g(x))$ for $x \geqq x_0 - g(x_0)$. By (1),

$$\Omega_n(x, g(x)) \leqq K_1^n \frac{[g(x)]^{\exp \omega_p\{[g(x)]^{-1-\epsilon}\}}}{[g(x)]^n} \qquad (K_1 = \text{const.} > 0).$$

For the sake of simplicity, consider the case $g(x) = e^{-x}$. We have

$$\Omega_n(x, g(x)) \leqq K_1^n (e^{-x \exp \omega_p(e^{(1+\epsilon)x})}) e^{nx}$$

for $x \geqq x_0 - e^{-x_0}$. The preceding corollary shows that

$$\Omega_n(x, g(x)) \leqq [K_1(\log n)(\log_2 n) \cdots (\log_p n)]^n$$

for integral $x \geqq x_0 - e^{-x_0}$. But, for $0 < \delta < 1$,

$$\frac{\exp\{-(x + \delta)'[\exp \omega_p(e^{(1+\epsilon)(x+\delta)})]\} e^{n(x+\delta)}}{\exp\{-x[\exp \omega_p(e^{(1+\epsilon)x})]\} e^{nx}} \leqq e^n \qquad (x \geqq 0).$$

Hence we obtain

$$\Omega_n(x, g(x)) \leqq [K_2(\log n)(\log_2 n) \cdots (\log_p n)]^n \quad (K_2 = \text{const.} > 0)$$

for $x \geqq x_0 - e^{-x_0}$ and for n sufficiently large. (We pass from the case $g(x) = e^{-x}$ to the general case simply by replacing e^{-x} in what precedes by $g(x)$.) Consequently we have

$$|F^{(n)}(x)| \leqq [K_3(\log n)(\log_2 n) \cdots (\log_p n)]^n \quad (K_3 = \text{const.} > 0)$$

for $x \geqq 0$ and for n sufficiently large. $F(x)$ and its derivatives of lower orders are evidently also bounded for $x \geqq 0$. An application of a Mandelbrojt's result on generalized quasi-analyticity [2, chap. III][4] will complete immediately the proof of our theorem.

From this theorem it follows that if $F_1(z)$ and $F_2(z)$ are functions holomorphic in Δ and verifying conditions similar to (4) and if $F_1^{(\nu_n)}(0) = F_2^{(\nu_n)}(0)$ for a sequence $\{\nu_n\}$ defined in the above theorem, then we have $F_1(z) \equiv F_2(z)$.

We remark that in the case $p=1$, (4) reduces to

$$F(z) = O\{[g(x)]^{\exp\{[g(x)]^{-1-\epsilon}\}}\} \qquad (z \text{ in } \Delta;\ x \to \infty). \tag{4}$$

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[4] For the case $p=1$ of the mentioned result, see [1, p. 372].