A REMARK ON KRONECKER'S THEOREM ON FORMS

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The Kronecker theorem on forms over an integral domain is a consequence of integral closure. We refer the reader to [1][1] for a proof and other references. We have failed to find in the literature a statement of the converse of this result and consequently shall here prove that integral closure is a consequence of a relatively weak form [2] of Kronecker's theorem.

Let $\mathfrak{O}$ be an integral domain which has the following property: Whenever each coefficient of the product of a linear polynomial $f(X)=a_0X+a_1$ of $\mathfrak{O}[X]$ by an arbitrary polynomial $g(X)=b_0X^n+\cdots+b_n$ of $\mathfrak{O}[X]$ is divisible by an element c of $\mathfrak{O}$, then a_1b_0 is divisible by c. Then $\mathfrak{O}$ is integrally closed in its quotient field k.

To prove this, we let $\alpha=u/v\in k$, u, $v\in\mathfrak{O}$, and assume α is integral over $\mathfrak{O}$. Thus $h(\alpha)=0$ where

$$h(X) = X^m + c_1X^{m-1} + \cdots + c_m$$

with all $c_j\in\mathfrak{O}$. Clearly

$$h(X) = (X - \alpha)(X^{m-1} + \beta_1X^{m-2} + \cdots + \beta_{m-1})$$

with $\beta_j\in k$. We select $w\in\mathfrak{O}$ such that $w\beta_j\in\mathfrak{O}$ for all j and have

$$vwh(X) = (vX - u)(wX^{m-1} + w\beta_1X^{m-2} + \cdots + w\beta_{m-1}).$$

Since vw divides each coefficient of the left-hand side of this equation, it follows from the hypothesis that vw divides uw, and hence v divides u. This implies that α is in $\mathfrak{O}$, which completes the proof.

References

1. W. Krull, *Idealtheorie*, Ergebnisse der Mathematik vol. 4, no. 3, 1935, especially §45.

2. H. Weyl, *Algebraic theory of numbers*, Princeton, 1940, especially Lemma II 7, A, p. 55.

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[1] Numbers in brackets refer to the references at the end of the paper.