$$(6.8) \qquad \overset{m}{\tau_r}(Y^{S^k}, Y_0^{S^k}, y_0) \simeq \overset{m+k}{\tau_r}(Y, Y_0, y_0) \otimes \overset{m}{\tau_r}(Y, Y_0, y_0).$$

The isomorphism (6.3) implies, with certain theorems given in [1], the interesting corollary that the Whitehead products of elements of the groups $\Pi_m(Y^{S^k}\{s_k, y_0\}, y_0)$, $k>0$, $m=1, 2, \cdots$, are all trivial. (6.1) and (6.2) show that the entire homotopy sequence may be reduced, one chunk at a time, to low-dimensional segments of homotopy sequences of suitable function spaces.

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A THEOREM ON INVOLUTORY TRANSFORMATIONS WITHOUT FIXED POINTS

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This note contains a proof of a theorem conjectured by Dr. Preston C. Hammer in his work on outwardly simple line families.

By E^{n+1} we denote the $(n+1)$-dimensional number space of ordered $(n+1)$-tuples of real numbers; if $X=(x_1, x_2, \cdots, x_{n+1})$, $Y=(y_1, y_2, \cdots, y_{n+1})$, and α, β are real numbers, we define as usual

$$\alpha X + \beta Y = (\alpha x_1 + \beta y_1, \cdots), \qquad (X, Y) = \sum_{i=1}^{n+1} x_i y_i.$$

By the n-sphere S^n we mean the set of all X in E^{n+1} such that $(X, X)=1$; a point X is said to be interior to S^n if $(X, X)<1$. A continuous map ϕ: $S^n \rightarrow S^n$ of the n-sphere into itself is called an involutory transformation if $\phi(\phi(Q))=Q$ for all Q on S^n.

Theorem. *Let ϕ: $S^n \rightarrow S^n$ be an involutory transformation of S^n without fixed point, and let P be a point interior to S^n. Then there is a point Q on S^n such that P lies on the line segment from Q to $\phi(Q)$.*

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The theorem is easily proved when n is even by using the fact that tangent vector fields cannot exist on even-dimensional spheres; the merit of the following proof is that it treats all dimensions similarly. The methods of the proof are similar to those P. A. Smith has developed to deal with periodic transformations.[1]

PROOF. Assume that the theorem is false. Then since $\phi(Q)\neq Q$, the points P, Q, $\phi(Q)$ determine a plane in E^{n+1}. We define $v(Q)$ to be a unit vector in the plane $PQ\phi(Q)$, perpendicular to the line $Q\phi(Q)$, and directed away from P, i.e.,

$$v(Q) = \rho\{Q - P - (Q - P, Q - \phi(Q))[Q - \phi(Q)]\}$$

where ρ is a normalizing factor. By its definition $v(Q)=v(\phi(Q))$; for all Q on S^n, $v(Q)\neq Q$, for if $v(Q)=Q$, then $Q\phi(Q)$ would be tangent to S^n which is manifestly impossible.

By the Lefschetz fixed point theorem, since $v(Q)\neq Q$, the topological degree of v must satisfy $1+(-1)^n \deg (v)=0$, or $\deg (v) =(-1)^{n+1}$.

We shall now construct a generator of $H_n(S^n, \text{mod } 2)$ of the form $c^n+\phi(c^n)$, so that $v(c^n+\phi(c^n))=2v(c^n)$ and the degree of v is even. This contradiction will establish the theorem.

Let us identify pairs Q, $\phi(Q)$ of points on S^n; by the assumptions on ϕ, the space Σ^n of such identified points is a manifold, and the natural map π: $S^n\to\Sigma^n$ defined by $\pi(Q)=\{Q, \phi(Q)\}$ is a covering map under which S^n covers Σ^n twice. Let d^n be a singular n-chain, which is a generator of $H_n(\Sigma^n, \text{mod } 2)$; further let the simplexes of the n-chain be so small that their inverse images under π are pairs of disjoint simplexes. Then $\pi^{-1}(d^n)$ is a generator of $H_n(S^n, \text{mod } 2)$ of the form $c^n+\phi(c^n)$.

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[1] See for example P. A. Smith, *Fixed points of periodic transformations*, in S. Lefschetz, *Algebraic topology*, Appendix II.